Introduction
class="introduction"

Purchasing pounds of fruit at a fruit market requires a basic understanding of numbers. (credit: Dr. Karl-Heinz Hochhaus, Wikimedia Commons)

Even though counting is first taught at a young age, mastering mathematics, which includes the study of numbers, requires constant attention. If it has been a while since you have studied math, it can be helpful to review basic topics. In this chapter, we will focus on numbers used for counting as well as four arithmetic operations—addition, subtraction, multiplication, and

division. We will also discuss some vocabulary that we will use throughout this book.

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Purchasing pounds of fruit at a fruit market requires a basic understanding of numbers. (credit: Dr. Karl-Heinz Hochhaus, Wikimedia Commons)

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We will also discuss some vocabulary that we will use throughout this book.

Introduction to Whole Numbers
By the end of this section, you will be able to:

- Contrast numbers, numerals, and digits
- Identify counting numbers and whole numbers
- Model whole numbers
- Identify the place value of a digit
- Use place value to name whole numbers
- Use place value to write whole numbers
- Round whole numbers

Contrast numbers, numerals, and digits

A **number** is a count or measure. It is an idea rather than a physical object.

A **numeral** is a name or symbol, or group of them, used to write or talk about numbers.

For example, when I want to refer to the idea of 5, I can write 5, or “five”, or if I’m using Roman numerals “V”. If I use a language other than English, the word or name for the idea 5 would be different. In Spanish, 5 is “cinco”. In Swahili, 5 is “tano”.

A **digit** is a single symbol used to refer to a number. The written text, 23, is a numeral, but it is not a digit because it uses two symbols, 2 and 3. 2 and 3 are each digits because they are single symbols that refer to a number.

Identify Counting Numbers and Whole Numbers

Learning algebra is similar to learning a language. You start with a basic vocabulary and then add to it as you go along. You need to practice often until the vocabulary becomes easy to you. The more you use the vocabulary, the more familiar it becomes.

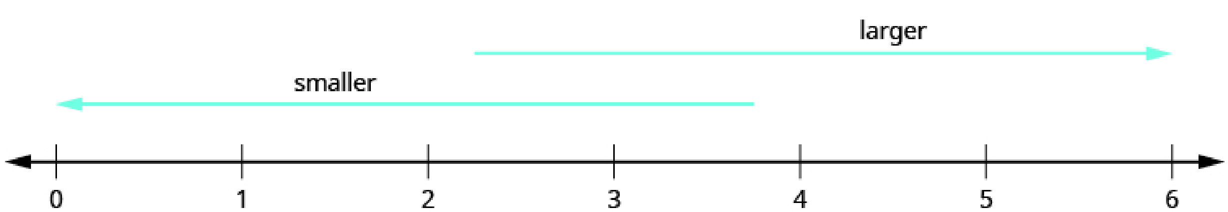
Algebra uses numbers and symbols to represent words and ideas. Let’s look at the numbers first. The most basic numbers used in algebra are those we use to count objects: $1, 2, 3, 4, 5, \ldots$ and so on. These are called the

counting numbers. The notation "…" is called an ellipsis, which is another way to show "and so on", or that the pattern continues endlessly. Counting numbers are also called natural numbers.

Note:
Counting Numbers
The counting numbers start with 1 and continue.
Equation:

$$1, 2, 3, 4, 5...$$

Counting numbers and whole numbers can be visualized on a **number line** as shown in [link].



The numbers on the number line increase from left to right, and decrease from right to left.

The point labeled 0 is called the **origin**. The distance between the origin and the point labeled "1" is one unit. The next counting number is one more unit to the right. This continues as much as needed. When a number is paired with a point on the number line, it is called the **coordinate** of the point.

The discovery of the number zero was a big step in the history of mathematics. Including zero with the counting numbers gives a new set of numbers called the **whole numbers**.

Note:
Whole Numbers
The whole numbers are the counting numbers and zero.
Equation:

$$0, 1, 2, 3, 4, 5 \ldots$$

We stopped at 5 when listing the first few counting numbers and whole numbers. We could have written more numbers if they were needed to make the patterns clear.

Example:
Exercise:

Problem:

Which of the following are ⓐ counting numbers? ⓑ whole numbers?

$0, \frac{1}{4}, 3, 5.2, 15, -7, 105$

Solution:
Solution

- ⓐ The counting numbers start at 1, so 0 is not a counting number. The numbers $3, 15,$ and 105 are all counting numbers.
- ⓑ Whole numbers are counting numbers and 0. The numbers $0, 3, 15,$ and 105 are whole numbers.

The numbers $\frac{1}{4}$, -7, and 5.2 are neither counting numbers nor whole numbers. We will discuss these numbers later.

Note:
Exercise:

Problem:

Which of the following are ⓐ counting numbers ⓑ whole numbers?

$0, \frac{2}{3}, 2, 9, 11.8, 241, 376$

Solution:

- ⓐ 2, 9, 241, 376
- ⓑ 0, 2, 9, 241, 376

Note:
Exercise:

Problem:

Which of the following are ⓐ counting numbers ⓑ whole numbers?

$0, \frac{5}{3}, 7, 8.8, 13, -17, 201$

Solution:

- ⓐ 7, 13, 201
- ⓑ 0, 7, 13, 201

Model Whole Numbers

Our number system is called a **place value system** because the value of a digit depends on its position, or place, in a number. The number 537 has a different value than the number 735. Even though they use the same digits, their value is different because of the different placement of the 3 and the 7 and the 5.

Money gives us a familiar model of place value. Suppose a wallet contains three \$100 bills, seven \$10 bills, and four \$1 bills. The amounts are summarized in [link]. How much money is in the wallet?

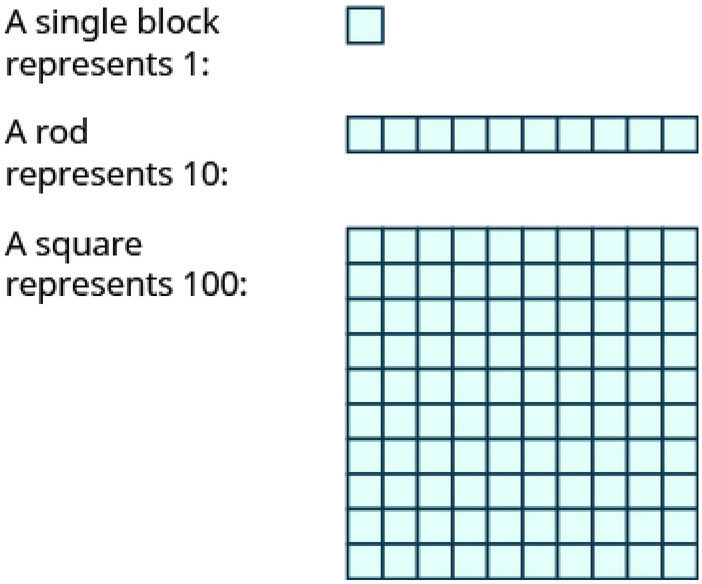
Three \$100 bills	Seven \$10 bills	Four \$1 bills
$3 \times \$100$	$7 \times \$10$	$4 \times \$1$
\$300	\$70	\$4

Find the total value of each kind of bill, and then add to find the total. The wallet contains \$374.

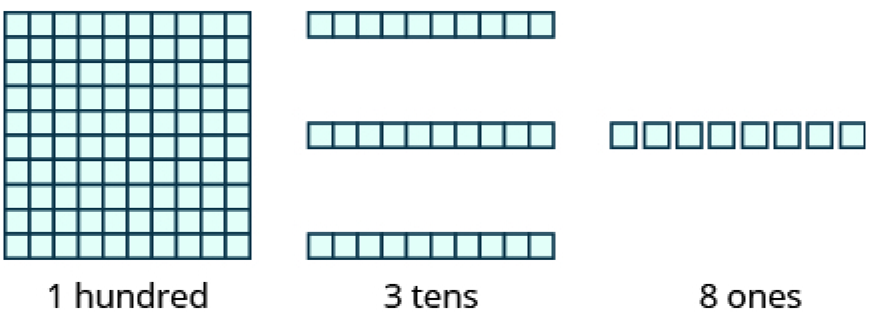
$$\$300 + \$70 + \$4$$

$$\$374$$

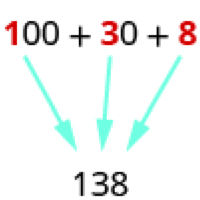
Base-10 blocks provide another way to model place value, as shown in [link]. The blocks can be used to represent hundreds, tens, and ones. Notice that the tens rod is made up of 10 ones, and the hundreds square is made of 10 tens, or 100 ones.



[link] shows the number 138 modeled with base-10 blocks.



We use place value notation to show the value of the number 138.



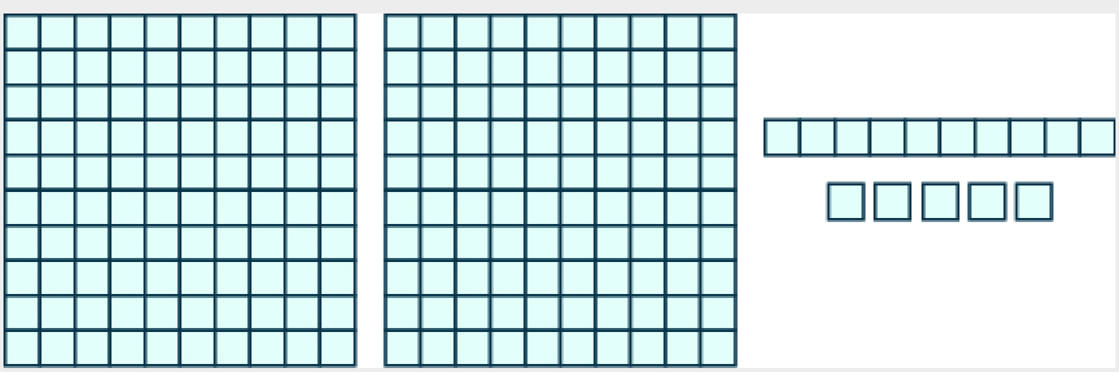
Digit	Place value	Number	Value	Total value
1	hundreds	1	100	100
3	tens	3	10	30
8	ones	8	1	$+\ 8$
				Sum $=138$

Expanded Form
Both money and base-10 blocks illustrate the **expanded form** of a number. The expanded form explicitly shows the value of each digit.
The expanded form of 374 is 300 + 70 + 4, and the expanded form of 138 is 100 + 30 + 8.
Standard form, also called **compact form**, is the way we normally write numbers: 374 and 138.

Example:
Exercise:

Problem:

Use place value notation and expanded form to find the value of the number modeled by the base-10 blocks shown.



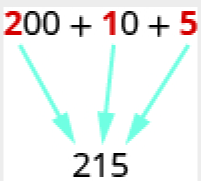
Solution:
Solution

There are 2 hundreds squares, which is 200.

There is 1 tens rod, which is 10.

There are 5 ones blocks, which is 5.

The expanded form is first followed by the standard form.



Digit	Place value	Number	Value	Total value
2	hundreds	2	100	200
1	tens	1	10	10
5	ones	5	1	+ 5
				215

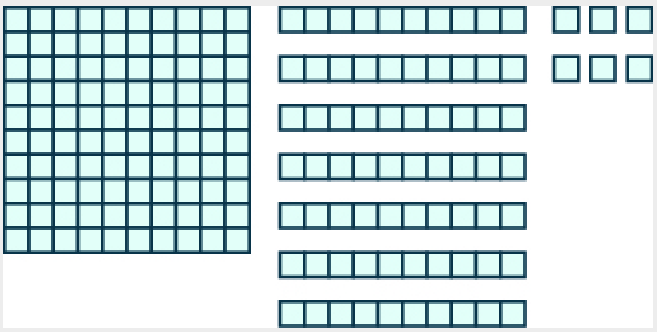
The base-10 blocks model the number 215.

Note:

Exercise:

Problem:

Use place value notation to find the expanded form and then the value of the number modeled by the base-10 blocks shown.



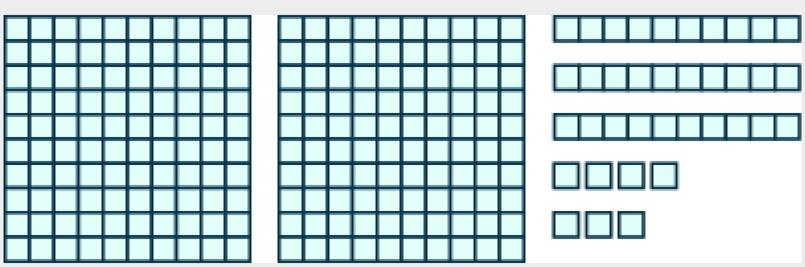
Solution:

Expanded form: 100 + 70 + 6
Value: 176

Note:
Exercise:

Problem:

Use place value notation to find expanded form and the value of the number modeled by the base-10 blocks shown.



Solution:

Expanded form: 200 + 30 + 7
Value: 237

Identify the Place Value of a Digit

By looking at money and base-10 blocks, we saw that each place in a number has a different value. A place value chart is a useful way to summarize this information. The place values are separated into groups of three, called periods. The periods are *ones*, *thousands*, *millions*, *billions*, *trillions*, and so on. In a written number, commas separate the periods.

Just as with the base-10 blocks, where the value of the tens rod is ten times the value of the ones block and the value of the hundreds square is ten times the tens rod, the value of each place in the place-value chart is ten times the value of the place to the right of it.

[link] shows how the number 5,278,194 is written in a place value chart.

Place Value														
Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
								5	2	7	8	1	9	4

- The digit 5 is in the millions place. Its value is 5,000,000.
- The digit 2 is in the hundred thousands place. Its value is 200,000.
- The digit 7 is in the ten thousands place. Its value is 70,000.
- The digit 8 is in the thousands place. Its value is 8,000.
- The digit 1 is in the hundreds place. Its value is 100.
- The digit 9 is in the tens place. Its value is 90.
- The digit 4 is in the ones place. Its value is 4.

Example:
Exercise:

Problem:

In the number 63,407,218; find the place value of each of the following digits:

ⓐ 7
ⓑ 0
ⓒ 1
ⓓ 6
ⓔ 3

Solution:

Solution

Write the number in a place value chart, starting at the right.

Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
							6	3	4	0	7	2	1	8

- ⓐ The 7 is in the thousands place.
- ⓑ The 0 is in the ten thousands place.
- ⓒ The 1 is in the tens place.
- ⓓ The 6 is in the ten millions place.
- ⓔ The 3 is in the millions place.

Note:

Exercise:

Problem:

For each number, find the place value of digits listed: 27,493,615

ⓐ 2

ⓑ 1
ⓒ 4
ⓓ 7
ⓔ 5

Solution:

ⓐ ten millions
ⓑ tens
ⓒ hundred thousands
ⓓ millions
ⓔ ones

Note:
Exercise:

Problem:

For each number, find the place value of digits listed:
519,711,641,328

ⓐ 9
ⓑ 4
ⓒ 2
ⓓ 6
ⓔ 7

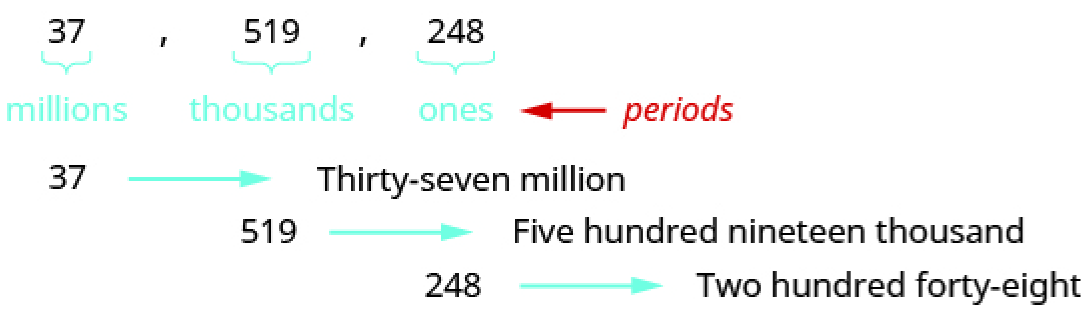
Solution:

ⓐ billions
ⓑ ten thousands
ⓒ tens
ⓓ hundred thousands

ⓔ hundred millions

Use Place Value to Name Whole Numbers

When you write a check, you write out the number in words as well as in digits. To write a number in words, write the number in each period followed by the name of the period without the ‘s’ at the end. Start with the digit at the left, which has the largest place value. The commas separate the periods, so wherever there is a comma in the number, write a comma between the words. The ones period, which has the smallest place value, is not named.



So the number 37,519,248 is written thirty-seven million, five hundred nineteen thousand, two hundred forty-eight.

Notice that the word *and* is not used when naming a whole number.

Note:
Name a whole number in words.

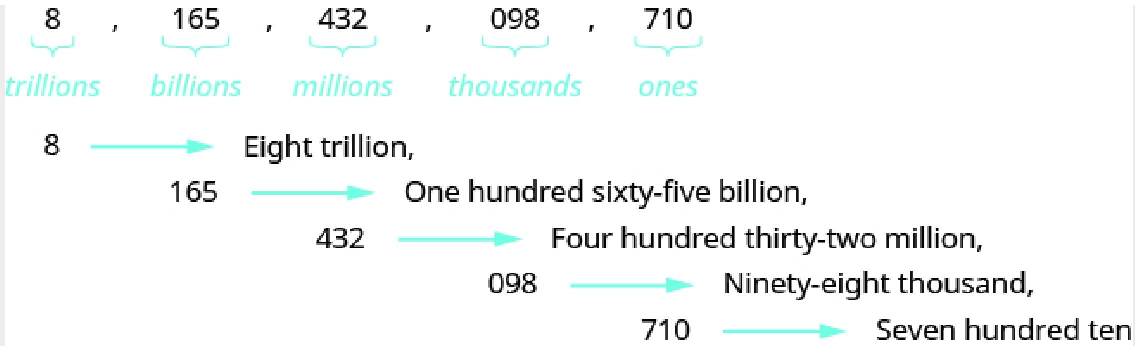
Starting at the digit on the left, name the number in each period, followed by the period name. Do not include the period name for the ones.
Use commas in the number to separate the periods.

Example:
Exercise:

Problem: Name the number 8,165,432,098,710 in words.

Solution:
Solution

Begin with the leftmost digit, which is 8. It is in the trillions place.	eight trillion
The next period to the right is billions.	one hundred sixty-five billion
The next period to the right is millions.	four hundred thirty-two million
The next period to the right is thousands.	ninety-eight thousand
The rightmost period shows the ones.	seven hundred ten



Putting all of the words together, we write 8,165,432,098,710 as eight trillion, one hundred sixty-five billion, four hundred thirty-two million, ninety-eight thousand, seven hundred ten.

A common error is to put the word "and" into the written whole number. For example, eight trillion, one hundred sixty-five billion, four hundred thirty-two million, ninety-eight thousand, seven hundred and ten. Even though most people will know what number you are trying to write, it isn't correct.

Note:
Exercise:

Problem: Name each number in words: 9,258,137,904,061

Solution:

nine trillion, two hundred fifty-eight billion, one hundred thirty-seven million, nine hundred four thousand, sixty-one

Note:
Exercise:

Problem: Name each number in words: 17,864,325,619,004

Solution:

seventeen trillion, eight hundred sixty-four billion, three hundred twenty-five million, six hundred nineteen thousand, four

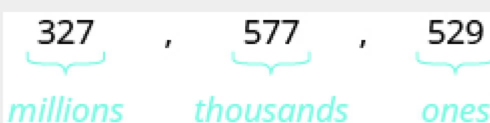
Example:
Exercise:

Problem:

A student conducted research and found that the number of mobile phone users in the United States during one month in 2014 was 327,577,529. Name that number in words.

Solution:
Solution

Identify the periods associated with the number.



Name the number in each period, followed by the period name. Put the commas in to separate the periods.

Millions period: three hundred twenty-seven million

Thousands period: five hundred seventy-seven thousand

Ones period: five hundred twenty-nine

So the number of mobile phone users in the Unites States during the month of April was three hundred twenty-seven million, five hundred seventy-seven thousand, five hundred twenty-nine.

Note:
Exercise:

Problem:

The population in a country is 316,128,839. Name that number.

Solution:

three hundred sixteen million, one hundred twenty-eight thousand, eight hundred thirty nine

Note:
Exercise:

Problem: One year is 31,536,000 seconds. Name that number.

Solution:

thirty one million, five hundred thirty-six thousand

Use Place Value to Write Whole Numbers

We will now reverse the process and write a number given in words as digits.

Note:
Use place value to write a whole number.

Identify the words that indicate periods. (Remember the ones period is never named.)

Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.
Name the number in each period and place the digits in the correct place value position.

Example:
Exercise:

Problem: Write the following numbers using digits.

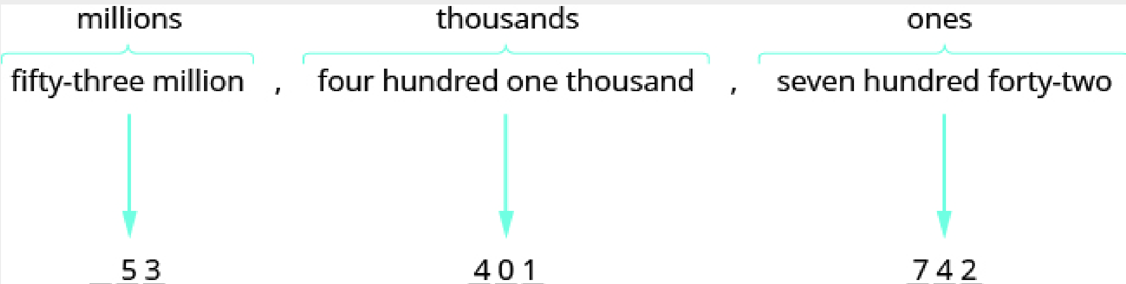
- ⓐ fifty-three million, four hundred one thousand, seven hundred forty-two
- ⓑ nine billion, two hundred forty-six million, seventy-three thousand, one hundred eighty-nine

Solution:
Solution

ⓐ Identify the words that indicate periods.

Except for the first period, all other periods must have three places. Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.

Then write the digits in each period.

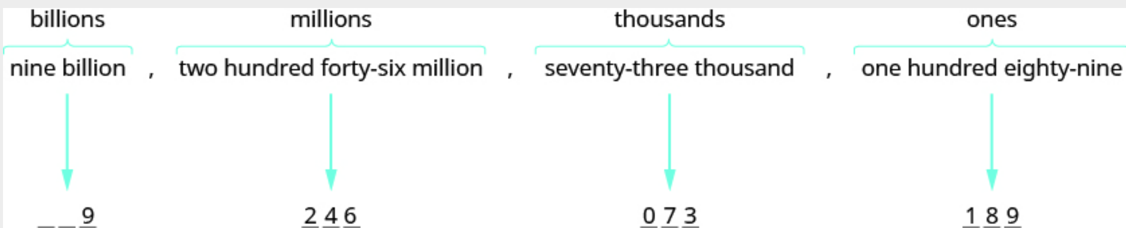


Put the numbers together, including the commas. The number is 53,401,742.

ⓑ Identify the words that indicate periods.

Except for the first period, all other periods must have three places. Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.

Then write the digits in each period.



The number is 9,246,073,189.

Notice that in part ⓑ, a zero was needed as a place-holder in the hundred thousands place. Be sure to write zeros as needed to make sure that each period, except possibly the first, has three places.

Note:
Exercise:

Problem: Write each number in standard form:

fifty-three million, eight hundred nine thousand, fifty-one.

Solution:

53,809,051

Note:
Exercise:

Problem: Write each number in standard form:

two billion, twenty-two million, seven hundred fourteen thousand, four hundred sixty-six.

Solution:

2,022,714,466

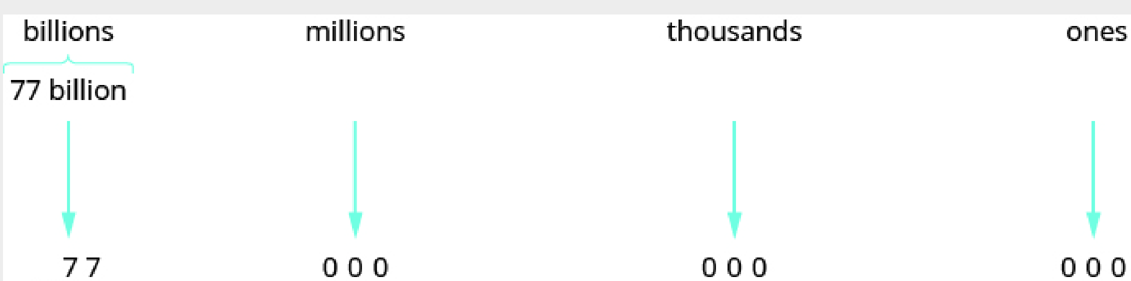
Example:
Exercise:

Problem:

A state budget was about \$77 billion. Write the budget in standard form.

Solution:
Solution

Identify the periods. In this case, only two digits are given and they are in the billions period. To write the entire number, write zeros for all of the other periods.



So the budget was about \$77,000,000,000.

Note:
Exercise:

Problem: Write each number in standard form:

The closest distance from Earth to Mars is about 34 million miles.

Solution:

34,000,000 miles

Note:
Exercise:

Problem: Write each number in standard form:

The total weight of an aircraft carrier is 204 million pounds.

Solution:

204,000,000 pounds

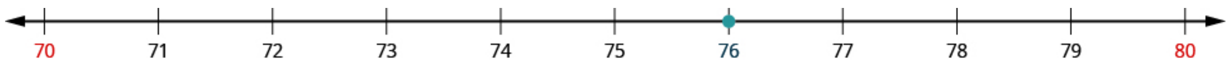
Round Whole Numbers

In 2018, the U.S. Census Bureau reported the population of the state of Illinois as 12,768,320 people. It might be enough to say that the population is approximately 13 million. The word *approximately* means that 13 million is not the exact population, but is close to the exact value.

The process of approximating a number is called **rounding**. Numbers are rounded to a specific place value depending on how much accuracy is needed. 13 million was achieved by rounding to the millions place. Had we

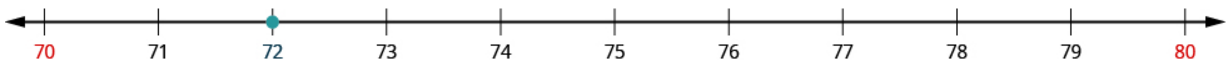
rounded to the one hundred thousands place, we would have 12,800,000 as a result. Had we rounded to the thousands place, we would have 12,768,000 as a result, and so on. The place value to which we round to depends on how we need to use the value.

Using the number line can help you visualize and understand the rounding process. Look at the number line in [link]. Suppose we want to round the number 76 to the nearest ten. Is 76 closer to 70 or 80 on the number line?



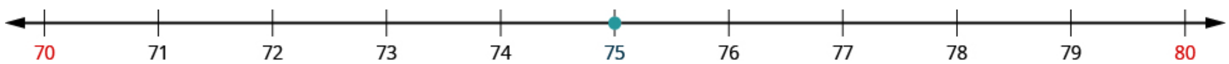
We can see that 76 is closer to 80 than to 70. So 76 rounded to the nearest ten is 80.

Now consider the number 72. Find 72 in [link].



We can see that 72 is closer to 70, so 72 rounded to the nearest ten is 70.

How do we round 75 to the nearest ten. Find 75 in [link].



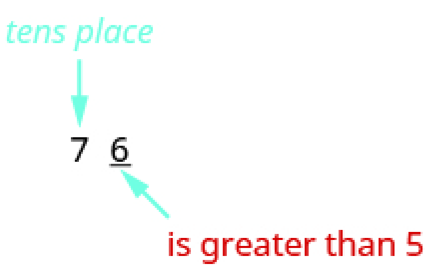
The number 75 is exactly midway between 70 and 80.

So that everyone rounds the same way in cases like this, mathematicians have agreed to round to the higher number, 80. So, 75 rounded to the nearest ten is 80.

Now that we have looked at this process on the number line, we can introduce a more general procedure. To round a number to a specific place, look at the number to the right of that place. If the number is less than 5, round down. If it is greater than or equal to 5, round up.

Notice that if mathematicians had decided to round 5 down rather than up, rounding would not be as easy. Consider the number 7,506 rounded to the thousands place. 7,500 is half way between 7,000 and 8,000 so 7,506 is closer to 8,000. Therefore 7,506 should be rounded to 8,000. Rounding this way, you can not tell if a number should be rounded up or down just by looking at the single digit to the right of the place that you are rounding to. But if we agree to always round up with a 5 then we can. This is because any digits to the right of the 5 can never make the value smaller.

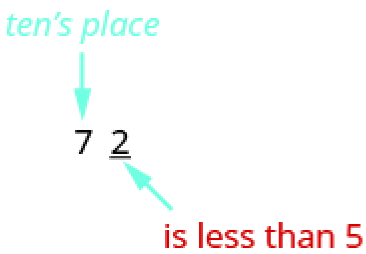
So, for example, to round 76 to the nearest ten, we look at the digit in the ones place.



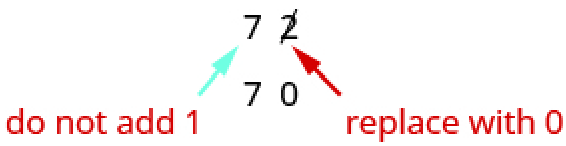
The digit in the ones place is a 6. Because 6 is greater than or equal to 5, we increase the digit in the tens place by one. So the 7 in the tens place becomes an 8. Now, replace any digits to the right of the 8 with zeros. So, 76 rounds to 80.

76 rounded to the nearest ten is 80.

Let’s look again at rounding 72 to the nearest 10. Again, we look to the ones place.



The digit in the ones place is 2. Because 2 is less than 5, we keep the digit in the tens place the same and replace the digits to the right of it with zero. So 72 rounded to the nearest ten is 70.



Note:
Round a whole number to a specific place value.

Locate the given place value. All digits to the left of that place value do not change.
Underline the digit to the right of the given place value.
Determine if this digit is greater than or equal to 5.

- Yes—add 1 to the digit in the given place value.
- No—do not change the digit in the given place value.

Replace all digits to the right of the given place value with zeros.

Example:
Exercise:

Problem: Round 843 to the nearest ten.

Solution:
Solution

Locate the tens place.	
Underline the digit to the right of the tens place.	$84\underline{3}$
Since 3 is less than 5, do not change the digit in the tens place.	$84\underline{3}$
Replace all digits to the right of the tens place with zeros.	$84\underline{0}$
	Rounding 843 to the nearest ten gives 840.

Note:
Exercise:

Problem: Round to the nearest ten: 157.

Solution:

160

Note:
Exercise:

Problem: Round to the nearest ten: 884.

Solution:

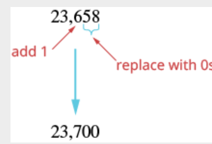
880

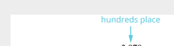
Example:
Exercise:

Problem: Round each number to the nearest hundred:

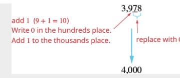
ⓐ $23,658$
ⓑ $3,978$

Solution:
Solution

ⓐ	
Locate the hundreds place.	
The digit of the right of the hundreds place is 5. Underline the digit to the right of the hundreds place.	$23{,}6\underline{5}8$
Since 5 is greater than or equal to 5, round up by adding 1 to the digit in the hundreds place. Then replace all digits to the right of the hundreds place with zeros.	 So 23,658 rounded to the nearest hundred is 23,700.

ⓑ	
Locate the hundreds place.	
Underline the digit to the right of the hundreds place.	$3{,}9\underline{7}8$
The digit to the right of the hundreds place is 7.	

Since 7 is greater than or equal to 5, round up by added 1 to the 9. Then place all digits to the right of the hundreds place with zeros.



So 3,978 rounded to the nearest hundred is 4,000.

Note:
Exercise:

Problem: Round to the nearest hundred: 17,852.

Solution:

17,900

Note:
Exercise:

Problem: Round to the nearest hundred: 4,951.

Solution:

5,000

Example:

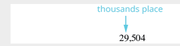
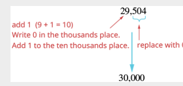
Exercise:

Problem: Round each number to the nearest thousand:

ⓐ 147,032
ⓑ 29,504

Solution:
Solution

ⓐ	
Locate the thousands place. Underline the digit to the right of the thousands place.	
The digit to the right of the thousands place is 0. Since 0 is less than 5, we do not change the digit in the thousands place.	147,032
We then replace all digits to the right of the thousands pace with zeros.	147,032 So 147,032 rounded to the nearest thousand is 147,000.

ⓑ	
Locate the thousands place.	
Underline the digit to the right of the thousands place.	29,504
The digit to the right of the thousands place is 5. Since 5 is greater than or equal to 5, round up by adding 1 to the 9. Then replace all digits to the right of the thousands place with zeros.	 So 29,504 rounded to the nearest thousand is 30,000.

Notice that in part ⓑ, when we add 1 thousand to the 9 thousands, the total is 10 thousands. We regroup this as 1 ten thousand and 0 thousands. We add the 1 ten thousand to the 3 ten thousands and put a 0 in the thousands place.

Note:
Exercise:

Problem: Round to the nearest thousand: 63,921.

Solution:

64,000

Note:
Exercise:

Problem: Round to the nearest thousand: $156,437$.

Solution:

156,000

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Determine Place Value
- Write a Whole Number in Digits from Words
- Contrast Numbers, Numerals, and Digits

Key Concepts

Place Value														
Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
								5	2	7	8	1	9	4

- **Name a whole number in words.**

  Starting at the digit on the left, name the number in each period, followed by the period name. Do not include the period name for the ones.
  Use commas in the number to separate the periods.

- **Use place value to write a whole number.**

  Identify the words that indicate periods. (Remember the ones period is never named.)
  Draw three blanks to indicate the number of places needed in each period.
  Name the number in each period and place the digits in the correct place value position.

- **Round a whole number to a specific place value.**

  Locate the given place value. All digits to the left of that place value do not change.
  Underline the digit to the right of the given place value.
  Determine if this digit is greater than or equal to 5. If yes—add 1 to the

digit in the given place value. If no—do not change the digit in the given place value.
Replace all digits to the right of the given place value with zeros.

Exercises

Practice Makes Perfect

Identify Counting Numbers and Whole Numbers

In the following exercises, determine which of the following numbers are ⓐ counting numbers ⓑ whole numbers.

Exercise:

Problem: $0, \frac{2}{3}, 5, 8.1, 125$

Solution:

ⓐ 5, 125
ⓑ 0, 5, 125

Exercise:

Problem: $0, \frac{7}{10}, 3, 20.5, 300$

Exercise:

Problem: $0, \frac{4}{9}, 3.9, 50, -120, 221$

Solution:

ⓐ 50, 221
ⓑ 0, 50, 221

Exercise:

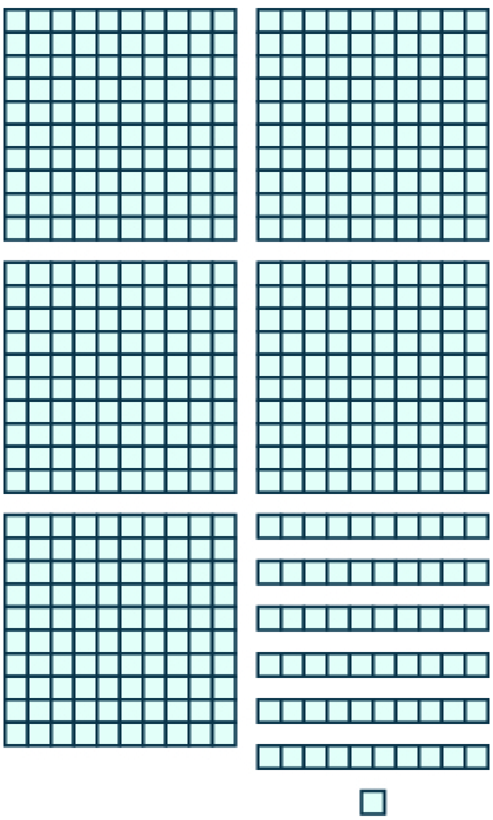
Problem: $0, \frac{3}{5}, 10, 303, 422.6$

Model Whole Numbers

In the following exercises, use place value notation to find the expanded form and the value of the number modeled by the base-10 blocks.

Exercise:

Problem:

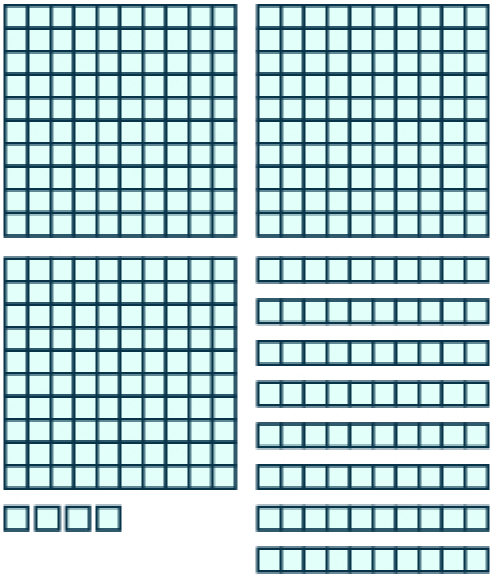


Solution:

Expanded form: 500 + 60 + 1
Value: 561

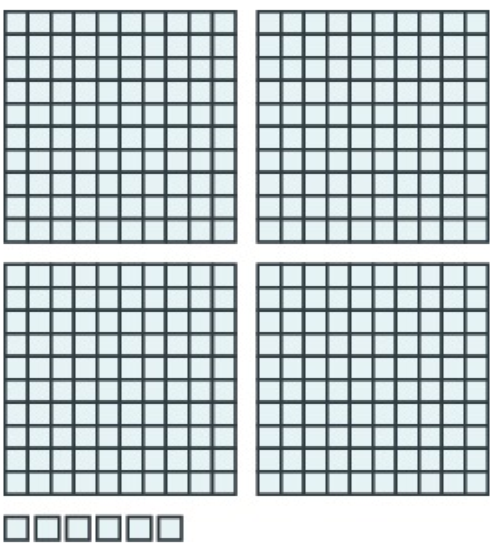
Exercise:

Problem:



Exercise:

Problem:

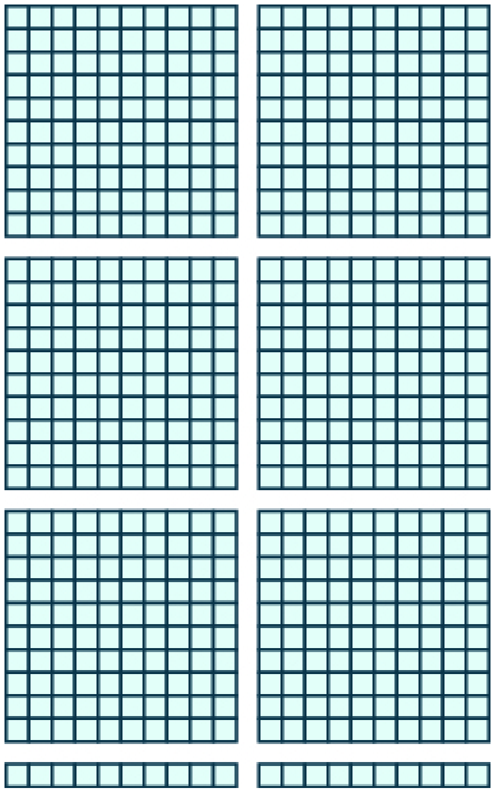


Solution:

Expanded form: 400 + 0 + 6
Value: 406

Exercise:

Problem:



Identify the Place Value of a Digit

In the following exercises, find the place value of the given digits.
Exercise:

Problem: $579{,}601$

ⓐ 9
ⓑ 6
ⓒ 0
ⓓ 7
ⓔ 5

Solution:

ⓐ thousands
ⓑ hundreds

ⓒ tens
ⓓ ten thousands
ⓔ hundred thousands

Exercise:

Problem: 398,127

ⓐ 9
ⓑ 3
ⓒ 2
ⓓ 8
ⓔ 7

Exercise:

Problem: 56,804,379

ⓐ 8
ⓑ 6
ⓒ 4
ⓓ 7
ⓔ 0

Solution:

ⓐ hundred thousands
ⓑ millions
ⓒ thousands
ⓓ tens
ⓔ ten thousands

Exercise:

Problem: 78,320,465

ⓐ 8
ⓑ 4
ⓒ 2
ⓓ 6
ⓔ 7

Use Place Value to Name Whole Numbers

In the following exercises, name each number in words.

Exercise:

Problem: 1,078

Solution:

One thousand, seventy-eight

Exercise:

Problem: 5,902

Exercise:

Problem: 364,510

Solution:

Three hundred sixty-four thousand, five hundred ten

Exercise:

Problem: 146,023

Exercise:

Problem: 5,846,103

Solution:

Five million, eight hundred forty-six thousand, one hundred three

Exercise:

Problem: 1,458,398

Exercise:

Problem: 37,889,005

Solution:

Thirty seven million, eight hundred eighty-nine thousand, five

Exercise:

Problem: 62,008,465

Exercise:

Problem: The height of Mount Ranier is 14,410 feet.

Solution:

Fourteen thousand, four hundred ten

Exercise:

Problem: The height of Mount Adams is 12,276 feet.

Exercise:

Problem: Seventy years is 613,200 hours.

Solution:

Six hundred thirteen thousand, two hundred

Exercise:

Problem: One year is $525,600$ minutes.

Exercise:

Problem:

The U.S. Census estimate of the population of Miami-Dade county was $2,617,176$.

Solution:

Two million, six hundred seventeen thousand, one hundred seventy-six

Exercise:

Problem: The population of Chicago was $2,718,782$.

Exercise:

Problem:

There are projected to be $23,867,000$ college and university students in the US in five years.

Solution:

Twenty three million, eight hundred sixty-seven thousand

Exercise:

Problem:

About twelve years ago there were $20,665,415$ registered automobiles in California.

Exercise:

Problem:

The population of China is expected to reach $1,377,583,156$ in 2016.

Solution:

One billion, three hundred seventy-seven million, five hundred eighty-three thousand, one hundred fifty-six

Exercise:

Problem:

The population of India is estimated at $1{,}267{,}401{,}849$ as of July $1, 2014$.

Use Place Value to Write Whole Numbers

In the following exercises, write each number as a whole number using digits.

Exercise:

Problem: four hundred twelve

Solution:

412

Exercise:

Problem: two hundred fifty-three

Exercise:

Problem: thirty-five thousand, nine hundred seventy-five

Solution:

35,975

Exercise:

Problem: sixty-one thousand, four hundred fifteen

Exercise:

Problem:

eleven million, forty-four thousand, one hundred sixty-seven

Solution:

11,044,167

Exercise:

Problem:

eighteen million, one hundred two thousand, seven hundred eighty-three

Exercise:

Problem:

three billion, two hundred twenty-six million, five hundred twelve thousand, seventeen

---

Solution:

3,226,512,017

Exercise:

Problem:

eleven billion, four hundred seventy-one million, thirty-six thousand, one hundred six

Exercise:

Problem:

The population of the world was estimated to be seven billion, one hundred seventy-three million people.

Solution:

7,173,000,000

Exercise:

Problem:

The age of the solar system is estimated to be four billion, five hundred sixty-eight million years.

Exercise:

Problem:

Lake Tahoe has a capacity of thirty-nine trillion gallons of water.

Solution:

39,000,000,000,000

Exercise:

Problem:

The federal government budget was three trillion, five hundred billion dollars.

Round Whole Numbers

In the following exercises, round to the indicated place value.

Exercise:

Problem: Round to the nearest ten:

ⓐ 386
ⓑ 2,931

Solution:

ⓐ 390
ⓑ 2,930

Exercise:

Problem: Round to the nearest ten:

ⓐ 792
ⓑ 5,647

Exercise:

Problem: Round to the nearest hundred:

ⓐ 13,748
ⓑ 391,794

Solution:

ⓐ 13,700
ⓑ 391,800

Exercise:

Problem: Round to the nearest hundred:

ⓐ 28,166
ⓑ 481,628

Exercise:

Problem: Round to the nearest ten:

ⓐ 1,492
ⓑ 1,497

Solution:

ⓐ 1,490
ⓑ 1,500

Exercise:

Problem: Round to the nearest thousand:

ⓐ 2,391
ⓑ 2,795

Exercise:

Problem: Round to the nearest hundred:

ⓐ 63,994
ⓑ 63,949

Solution:

ⓐ 64,000
ⓑ 63,900

Exercise:

Problem: Round to the nearest thousand:

ⓐ 163,584
ⓑ 163,246

Everyday Math

Exercise:

Problem:

Writing a Check Jorge bought a car for $24,493. He paid for the car with a check. Write the purchase price in words.

Solution:

Twenty four thousand, four hundred ninety-three dollars

Exercise:

Problem:

Writing a Check Marissa’s kitchen remodeling cost $18,549. She wrote a check to the contractor. Write the amount paid in words.

Exercise:

Problem:

Buying a Car Jorge bought a car for $24,493. Round the price to the nearest:

ⓐ ten dollars
ⓑ hundred dollars
ⓒ thousand dollars
ⓓ ten-thousand dollars

Solution:

ⓐ \$24,490
ⓑ \$24,500
ⓒ \$24,000
ⓓ \$20,000

Exercise:

Problem:

Remodeling a Kitchen Marissa's kitchen remodeling cost $\$18,549$. Round the cost to the nearest:

ⓐ ten dollars
ⓑ hundred dollars
ⓒ thousand dollars
ⓓ ten-thousand dollars

Exercise:

Problem:

Population The population of China was $1,355,692,544$ in 2014. Round the population to the nearest:

ⓐ billion people
ⓑ hundred-million people
ⓒ million people

Solution:

ⓐ $1,000,000,000$
ⓑ $1,400,000,000$
ⓒ $1,356,000,000$

Exercise:

Problem:

Astronomy The average distance between Earth and the sun is 149,597,888 kilometers. Round the distance to the nearest:

ⓐ hundred-million kilometers
ⓑ ten-million kilometers
ⓒ million kilometers

Writing Exercises

Exercise:

Problem:

In your own words, explain the difference between the counting numbers and the whole numbers.

Solution:

Answers may vary. The whole numbers are the counting numbers with the inclusion of zero.

Exercise:

Problem:

Give an example from your everyday life where it helps to round numbers.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
contrast numbers, numerals, and digits.			
identify counting numbers and whole numbers.			
model whole numbers.			
identify the place value of a digit.			
use place value to name whole numbers.			
use place value to write whole numbers.			
round whole numbers.			

ⓑ If most of your checks were...

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

compact form
: Same as standard form. The normal way we write whole numbers. Contrast with expanded form.

coordinate

A number paired with a point on a number line is called the coordinate of the point.

counting numbers
: The counting numbers are the numbers 1, 2, 3, ….

expanded form
: The expanded form explicitly shows the value of each digit.

number line
: A number line is used to visualize numbers. The numbers on the number line get larger as they go from left to right, and smaller as they go from right to left.

origin
: The origin is the point labeled 0 on a number line.

place value system
: Our number system is called a place value system because the value of a digit depends on its position, or place, in a number.

rounding
: The process of approximating a number is called rounding.

standard form
: Same as compact form.

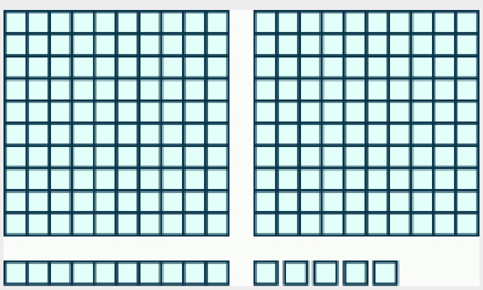
whole numbers
: The whole numbers are the numbers 0, 1, 2, 3, ….

Add Whole Numbers
By the end of this section, you will be able to:

- Use addition notation
- Model addition of whole numbers
- Add whole numbers without models
- Explain when and how addition can be more efficient than counting
- Recognize and explain the commutative and associative properties of addition
- Translate word phrases to math notation
- Add whole numbers in applications

Note:
Before you get started, take this readiness quiz.

1. What is the number modeled by the base-10 blocks?

   If you missed this problem, review [link].
2. Write the number three hundred forty-two thousand six using digits.
   If you missed this problem, review [link].

Use Addition Notation

A college student has a part-time job. Last week he worked 3 hours on Monday and 4 hours on Friday. To find the total number of hours he worked last week, he added 3 and 4.

The operation of addition combines numbers to get a **sum**. The notation we use to find the sum of 3 and 4 is:

Equation:

$$3 + 4$$

We read this as *three plus four* and the result is the sum of three and four. The numbers 3 and 4 are called the **addends**. A math statement that includes numbers and operations is called an **expression**.

Note:
Addition Notation
To describe addition, we can use symbols and words.

Operation	Notation	Expression	Read as	Result
Addition	$+$	$3 + 4$	three plus four	the sum of 3 and 4

Example:
Exercise:

Problem: Translate from math notation to words:

ⓐ $7 + 1$
ⓑ $12 + 14$

Solution:
Solution

- ⓐ The expression consists of a plus symbol connecting the addends 7 and 1. We read this as *seven plus one*. The result is *the sum of seven and*

one.

- ⓑ The expression consists of a plus symbol connecting the addends 12 and 14. We read this as *twelve plus fourteen*. The result is the *sum of twelve and fourteen*.

Note:
Exercise:

Problem: Translate from math notation to words:

ⓐ $8+4$
ⓑ $18+11$

Solution:

- ⓐ eight plus four; the sum of eight and four
- ⓑ eighteen plus eleven; the sum of eighteen and eleven

Note:
Exercise:

Problem: Translate from math notation to words:

ⓐ $21+16$
ⓑ $100+200$

Solution:

ⓐ twenty-one plus sixteen; the sum of twenty-one and sixteen
ⓑ one hundred plus two hundred; the sum of one hundred and two hundred

Counting versus Adding

Counting and adding are both used to determine an amount. Counting a small amount normally takes a short time, but counting a large amount can take a long time. For example, counting the number of students in your classroom probably takes at most a minute while counting the number of students in the entire building will take much longer.

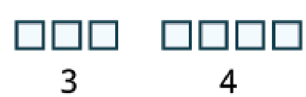
Addition can be more efficient than counting but that requires that the addends are already known. If I already have the number of students in each class, then it will be more efficient to add those values to find the number of students in the entire building than it would be to count all of the students directly. Getting those original values could have been done by previous counting, and adding makes use of that work. Simply recounting all of the students would be inefficient.

In order for addition to be most useful, one needs to know the addition facts and how to add.

Model Addition of Whole Numbers

Addition is often used to determine the amount after we put two different groups together. We will model addition with base-10 blocks. Remember, a block represents 1 and a rod represents 10. Let's start by modeling the addition expression we just considered, $3 + 4$.

Each addend is less than 10, so we can use ones blocks.

We start by modeling the first number with 3 blocks.	
Then we model the second number with 4 blocks.	

Put the parts together and count the total number of blocks.	

There are 7 blocks in all. We use an equal sign $(=)$ to show the sum. A math sentence that shows that two expressions are equal is called an equation. We have shown that. $3+4=7$.

Most people don't need to count the total number of blocks to know that there are seven of them. They know their addition facts and automatically recall that 3 + 4 = 7. When numbers are small like 3 and 4, it doesn't save much time to add rather than count. For large numbers, it makes a big difference.

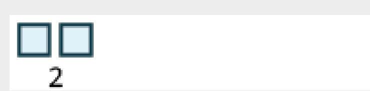
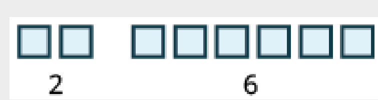
Example:
Exercise:

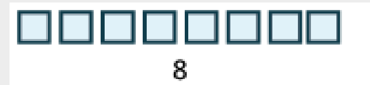
Problem: Model the addition $2+6$.

Solution:
Solution

$2+6$ means the sum of 2 and 6

Each addend is less than 10, so we can use ones blocks.

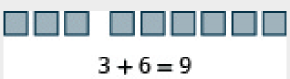
Model the first number with 2 blocks.	
Model the second number with 6 blocks.	

Put the parts together and count the total number of blocks	 There are 8 blocks in all, so $2+6=8$.

Note:
Exercise:

Problem: Model: $3+6$.

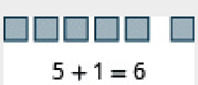
Solution:



Note:
Exercise:

Problem: Model: $5+1$.

Solution:



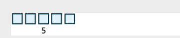
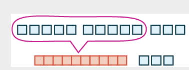
When the result is 10 or more ones blocks, we will exchange the 10 unit blocks for one rod.

Example:
Exercise:

Problem: Model the addition $5 + 8$.

Solution:
Solution

$5 + 8$ means the sum of 5 and 8.

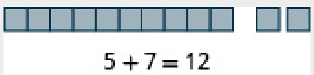
Each addend is less than 10, se we can use ones blocks.	
Model the first number with 5 blocks.	
Model the second number with 8 blocks.	
Count the result. There are more than 10 blocks so we exchange 10 ones blocks for 1 tens rod. We want to use the minimum number of pieces possible.	
Now we have 1 ten and 3 ones, which is 13.	5 + 8 = 13

Notice that we can describe the models as ones blocks and tens rods, or we can simply say *ones* and *tens*. From now on, we will use the shorter version but keep in mind that they mean the same thing.

Note:
Exercise:

Problem: Model the addition: $5+7$.

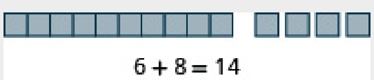
Solution:



Note:
Exercise:

Problem: Model the addition: $6+8$.

Solution:



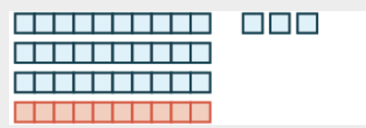
Next we will model adding two digit numbers.

Example:
Exercise:

Problem: Model the addition: $17+26$.

Solution:
Solution

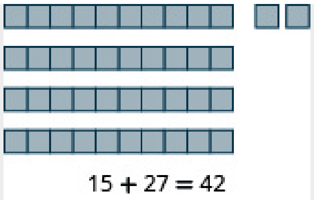
$17 + 26$ means the sum of 17 and 26.

Model the 17.	1 ten and 7 ones	
Model the 26.	2 tens and 6 ones	
Combine.	3 tens and 13 ones	
Exchange 10 ones for 1 ten.	4 tens and 3 ones $40 + 3 = 43$	
We have shown that $17 + 26 = 43$		

Note:
Exercise:

Problem: Model each addition: $15 + 27$.

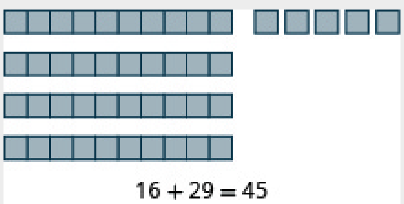
Solution:



Note:
Exercise:

Problem: Model each addition: $16 + 29$.

Solution:



Add Whole Numbers Without Models

Now that we have used models to add numbers, we can move on to adding without models. Before we do that, make sure you know all the one digit addition facts. You will need to use these number facts when you add larger numbers.

Imagine filling in [link] by adding each row number along the left side to each column number across the top. Make sure that you get each sum shown. If you have trouble, model it. It is important that you memorize any number facts you do not already know so that you can quickly and reliably use the number facts when you add larger numbers.

+	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Did you notice what happens when you add zero to a number? The sum of any number and zero is the number itself. We call this the Identity Property of Addition. Zero is called the additive identity.

Note:
Identity Property of Addition
The sum of any number a and 0 is the number.
Equation:

$$a + 0 = a$$
$$0 + a = a$$

Frequently in algebra, we use a letter to stand for potentially any number. When used that way, that letter is a **variable**.

Each of these equations has only one variable, the letter “a”. The “a” appears twice in each equation. For either of these equations, once we pick a value for “a” we have to use the same value for “a” everywhere in the equation. If we wanted to be able to pick a different value for the second instance of the variable, we would have had to use a different variable, perhaps the letter “b” although any other letter would do.

Example:
Exercise:

Problem:

Find each sum and explain how the Identity Property of Addition applies:

ⓐ $0+11$
ⓑ $42+0$

Solution:
Solution

ⓐ The first addend is zero. The sum of zero and any number is the number. In the equation 0 + a = a, if a = 11 then 0 + 11 = 11. Notice that a = 11 in both places.	$0+11=11$
ⓑ The second addend is zero. The sum of any number and zero is the number. In the equation a + 0 = a, if a = 42 then 42 + 0 = 42. Notice that a = 42 in both places.	$42+0=42$

Note:
Exercise:

Problem: Find each sum:

ⓐ $0 + 19$
ⓑ $39 + 0$

Solution:

ⓐ $0 + 19 = 19$
ⓑ $39 + 0 = 39$

Note:
Exercise:

Problem: Find each sum:

ⓐ $0 + 24$
ⓑ $57 + 0$

Solution:

ⓐ $0 + 24 = 24$
ⓑ $57 + 0 = 57$

Look at the pairs of sums.

$2 + 3 = 5$	$3 + 2 = 5$
$4 + 7 = 11$	$7 + 4 = 11$
$8 + 9 = 17$	$9 + 8 = 17$

Notice that when the order of the addends is reversed, the sum does not change. This property is called the Commutative Property of Addition, which states that changing the order of the addends does not change their sum. Notice that this property requires two variables.

Note:
Commutative Property of Addition
Changing the order of the addends a and b does not change their sum.
Equation:

$$a + b = b + a$$

Example:
Exercise:

Problem: Add:

ⓐ $8 + 7$
ⓑ $7 + 8$

Solution:
Solution

-

ⓐ	

Add.	$8 + 7$
	15

ⓑ	
Add.	$7 + 8$
	15

Did you notice that changing the order of the addends did not change their sum? We could have immediately known the sum from part ⓑ just by recognizing that the addends were the same as in part ⓑ, but in the reverse order. As a result, both sums are the same.

An equation that shows this instance of the Commutative Property of Addition is 8 + 7 = 7 + 8.

Note:
Exercise:

Problem: Add: $9 + 7$ and $7 + 9$.

Solution:

$9 + 7 = 16; 7 + 9 = 16$

Note:
Exercise:

Problem: Add: $8 + 6$ and $6 + 8$.

Solution:

$8 + 6 = 14; 6 + 8 = 14$

Example:
Exercise:

Problem: Add: $28 + 61$.

Solution:
Solution

To add numbers with more than one digit, it is often easier to write the numbers vertically in columns.

Write the numbers so the ones and tens digits line up vertically.	$\begin{array}{r} 28 \\ +61 \\ \hline \end{array}$
Then add the digits in each place value. Add the ones: $8 + 1 = 9$ Add the tens: $2 + 6 = 8$	$\begin{array}{r} 28 \\ +61 \\ \hline 89 \end{array}$

Summing 28 with 61 was easy. We did not need any addition facts beyond the single digit facts even though both of our addends were greater than 9. We were able to add each place separately. Without this, we would have too many addition facts to learn and we might be better off counting. By grouping large numbers using place value, we can use the same single digit facts repeatedly

so that there is only a small number of facts to learn. This helps adding be much more efficient than counting.

Why do we ever bother to count rather than add? Adding requires that we know the addends. In a textbook problem, the addends are usually given, but in everyday life, we are often not that lucky. Counting may be the only way to find the addends.

Note:
Exercise:

Problem: Add: $32 + 54$.

Solution:

$32 + 54 = 86$

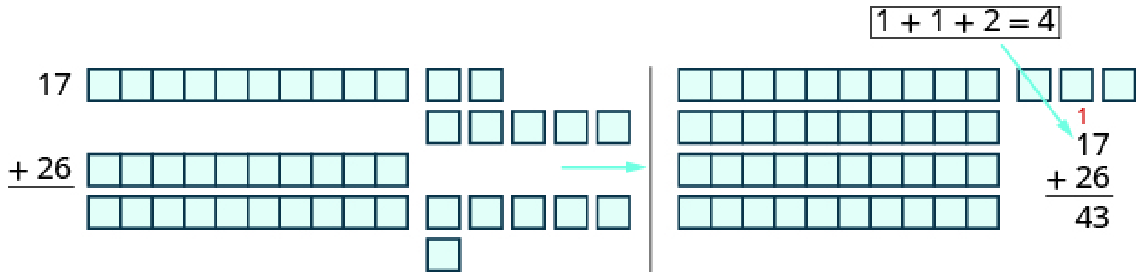
Note:
Exercise:

Problem: Add: $25 + 74$.

Solution:

$25 + 74 = 99$

In the previous example, the sum of the ones and the sum of the tens were both less than 10. But what happens if the sum is 10 or more? Let's use our base-10 model to find out. [link] shows the addition of 17 and 26 again.



When we add the ones, $7+6$, we get 13 ones. Because we have more than 10 ones, we can exchange 10 of the ones for 1 ten. Now we have 4 tens and 3 ones. Without using the model, we show this as a small red 1 above the digits in the tens place.

When the sum in a place value column is greater than 9, we carry over to the next column to the left. Carrying is the same as regrouping by exchanging. For example, 10 ones for 1 ten or 10 tens for 1 hundred.

Note:
Add whole numbers.

Write the numbers so each place value lines up vertically.
Add the digits in each place value. Work from right to left starting with the ones place. If a sum in a place value is more than 9, carry to the next place value.
Continue adding each place value from right to left, adding each place value and carrying if needed.

Why do we add by starting with on the right (the ones' place) and then move place by place to the left? Why not the other way around? The digits to the left are worth more than the digits to the right. Intuitively, is makes sense to start on the left side. When an approximate answer is likely to be all we need, this may be good enough. This is similar to rounding. If we need an exact answer, consider 17 + 26 again. If we add the tens place digits, we get 1 + 2 = 3. If we add the ones place digits, we get 7 + 6 = 13. If we have already written down the 3 in the tens' place, we would have to change it to a 4 because of the regrouping (carrying) from the ones' place. This is inefficient, and that is why it is better to start at the ones' place and move to the left.

Example:
Exercise:

Problem: Add: $43 + 69$.

Solution:
Solution

Write the numbers so the digits line up vertically.	$\begin{array}{r} 43 \\ +69 \\ \hline \end{array}$
Add the digits in each place. Add the ones: $3 + 9 = 12$	
Write the 2 in the ones place in the sum. Add the 1 ten to the tens place.	$\begin{array}{r} \overset{1}{4}3 \\ +69 \\ \hline 2 \end{array}$
Now add the tens: $1 + 4 + 6 = 11$ Write the 11 in the sum.	$\begin{array}{r} \overset{1}{4}3 \\ +69 \\ \hline 112 \end{array}$

Note:
Exercise:

Problem: Add: $35 + 98$.

Solution:

$35 + 98 = 133$

Note:
Exercise:

Problem: Add: $72 + 89$.

Solution:

$72 + 89 = 161$

Example:
Exercise:

Problem: Add: $324 + 586$.

Solution:
Solution

Write the numbers so the digits line up vertically.	$\begin{array}{r} 324 \\ +\ 586 \\ \hline \end{array}$
Add the digits in each place value. Add the ones: $4 + 6 = 10$ Write the 0 in the ones place in the sum and carry the 1 ten to the tens place.	$\begin{array}{r} 3\overset{1}{2}4 \\ +\ 586 \\ \hline 0 \end{array}$
Add the tens: $1 + 2 + 8 = 11$ Write the 1 in the tens place in the sum and carry the 1 hundred to the hundreds	$\begin{array}{r} 3\overset{1}{2}4 \\ +\ 586 \\ \hline 0 \end{array}$

Add the hundreds: $1 + 3 + 5 = 9$ Write the 9 in the hundreds place.	$\begin{array}{r} \overset{1}{} \\ 3\,2\,4 \\ +\;\;5\,8\,6 \\ \hline 0 \end{array}$

Note:
Exercise:

Problem: Add: $456 + 376$.

Solution:

$456 + 376 = 832$

Note:
Exercise:

Problem: Add: $269 + 578$.

Solution:

$269 + 578 = 847$

Example:
Exercise:

Problem: Add: $1{,}683 + 479$.

Solution:
Solution

Write the numbers so the digits line up vertically.	$\begin{array}{r} 1{,}683 \\ +\ 479 \\ \hline \end{array}$
Add the digits in each place value.	
Add the ones: $3 + 9 = 12$. Write the 2 in the ones place of the sum and carry the 1 ten to the tens place.	$\begin{array}{r} \scriptsize{1}\ \ \\ 1{,}683 \\ +\ 479 \\ \hline 2 \end{array}$
Add the tens: $1 + 7 + 8 = 16$ Write the 6 in the tens place and carry the 1 hundred to the hundreds place.	$\begin{array}{r} \scriptsize{1\ 1}\ \ \\ 1{,}683 \\ +\ 479 \\ \hline 62 \end{array}$
Add the hundreds: $1 + 6 + 4 = 11$ Write the 1 in the hundreds place and carry the 1 thousand to the thousands place.	$\begin{array}{r} \scriptsize{1\ 1}\ \ \\ 1{,}683 \\ +\ 479 \\ \hline 162 \end{array}$
Add the thousands $1 + 1 = 2$. Write the 2 in the thousands place of the sum.	$\begin{array}{r} \scriptsize{1\ 1}\ \ \\ 1{,}683 \\ +\ 479 \\ \hline 2{,}162 \end{array}$

When the addends have different numbers of digits, be careful to line up the corresponding place values starting with the ones and moving toward the left.

Note:
Exercise:

Problem: Add: $4{,}597 + 685$.

Solution:

$4{,}597 + 685 = 5{,}282$

Note:
Exercise:

Problem: Add: $5{,}837 + 695.$

Solution:

$5{,}837 + 695 = 6{,}532$

Example:
Exercise:

Problem: Add: $21{,}357 + 861 + 8{,}596.$

Solution:
Solution

Write the numbers so the place values line up vertically.	$\begin{array}{r} 21{,}357 \\ 861 \\ \underline{+\ 8{,}596} \end{array}$
Add the digits in each place value.	
Add the ones: $7 + 1 + 6 = 14$ Write the 4 in the ones place of the sum and carry the 1 to the tens place.	

	$\begin{array}{r} 21,3\overset{1}{5}7 \\ 861 \\ +\ 8,596 \\ \hline 4 \end{array}$
Add the tens: $1 + 5 + 6 + 9 = 21$ Write the 1 in the tens place and carry the 2 to the hundreds place.	$\begin{array}{r} 21,\overset{2}{3}\overset{1}{5}7 \\ 861 \\ +\ 8,596 \\ \hline 14 \end{array}$
Add the hundreds: $2 + 3 + 8 + 5 = 18$ Write the 8 in the hundreds place and carry the 1 to the thousands place.	$\begin{array}{r} 2\overset{1}{1},\overset{2}{3}\overset{1}{5}7 \\ 861 \\ +\ 8,596 \\ \hline 814 \end{array}$
Add the thousands $1 + 1 + 8 = 10$. Write the 0 in the thousands place and carry the 1 to the ten thousands place.	$\begin{array}{r} \overset{1}{2}\overset{1}{1},\overset{2}{3}\overset{1}{5}7 \\ 861 \\ +\ 8,596 \\ \hline 0814 \end{array}$
Add the ten-thousands $1 + 2 = 3$. Write the 3 in the ten thousands place in the sum.	$\begin{array}{r} \overset{1}{2}\overset{1}{1},\overset{2}{3}\overset{1}{5}7 \\ 861 \\ +\ 8,596 \\ \hline 30,814 \end{array}$

This example had three addends. We can add any number of addends using the same process as long as we are careful to line up the place values correctly.

Note:
Exercise:

Problem: Add: $46{,}195 + 397 + 6{,}281$.

Solution:

$46{,}195 + 397 + 6{,}281 = 52{,}873$

Note:
Exercise:

Problem: Add: $53{,}762 + 196 + 7{,}458$.

Solution:

$53{,}762 + 196 + 7{,}458 = 61{,}416$

Translate Word Phrases to Math Notation

Earlier in this section, we translated math notation into words. Now we'll reverse the process. We'll translate word phrases into math notation. Some of the word phrases that indicate addition are listed in [link].

Operation	Words	Example	Expression

Operation	Words	Example	Expression
Addition	plus sum increased by more than total of added to	1 plus 2 the sum of 3 and 4 5 increased by 6 8 more than 7 the total of 9 and 5 6 added to 4	$1+2$ $3+4$ $5+6$ $7+8$ $9+5$ $4+6$

Example:
Exercise:

Problem: Translate and simplify: the sum of 19 and 23.

Solution:
Solution

The word *sum* tells us to add. The words *of* 19 *and* 23 tell us the addends.

	The sum of 19 and 23
Translate.	$19+23$
Add.	42
	The sum of 19 and 23 is 42.

Note:
Exercise:

Problem: Translate and simplify: the sum of 17 and 26.

Solution:

Translate: $17+26$; Simplify: 43

Note:
Exercise:

Problem: Translate and simplify: the sum of 28 and 14.

Solution:

Translate: $28+14$; Simplify: 42

Example:
Exercise:

Problem: Translate and simplify: 28 increased by 31.

Solution:
Solution

The words *increased by* tell us to add. The numbers given are the addends.

	28 increased by 31.
Translate.	$28+31$
Add.	59

	So 28 increased by 31 is 59.

Note:
Exercise:

Problem: Translate and simplify: 29 increased by 76.

Solution:

Translate: $29 + 76$; Simplify 105

Note:
Exercise:

Problem: Translate and simplify: 37 increased by 69.

Solution:

Translate 37 + 69; Simplify 106

Add Whole Numbers in Applications

Plan for Solving Real-World Problems

- Read the problem to determine what we are looking for.
- Write a word phrase that gives the information to find it.
- Translate the word phrase into math notation.
- Simplify.
- Write a sentence to answer the question.

Example:
Exercise:

Problem:

Hao earned grades of $87, 91, 83, 82,$ and 89 on the five tests of the semester. What is the total number of points he earned on the five tests?

Solution:
Solution

We are asked to find the total number of points on the tests.

Write a phrase.	the sum of points on the tests
Translate to math notation.	$87 + 91 + 83 + 82 + 89$
Then we simplify by adding.	
Since there are several numbers, we will write them vertically.	$\begin{array}{r} \overset{2}{8}7 \\ 91 \\ 83 \\ 82 \\ \underline{+89} \\ 432 \end{array}$
Write a sentence to answer the question.	Hao earned a total of 432 points.

Notice that we added *points*, so the sum is 432 *points*. It is important to include the appropriate units in all answers to applications problems.

The Associative Property of Addition

Addition is so familiar to us that we may use properties that addition has without even realizing it.
In a previous problem we just calculated 87 + 91 + 83 + 82 + 89. Since there aren't any we parenthesis, we should add from left to right. That means we should have done 87 + 91 = 178 first. Then we should have taken that sum and added the next number: 178 + 83 = 261, and continued that way until finished eventually getting 432. But that probably isn't the way you did the problem. You probably added all of the ones place first and regrouped (carried) into the tens place and then added that column. Why do we get the same answer either way?

When we add column by column, even for just two numbers, we are actually breaking those numbers apart using expanded form, rearranging the addends, and then finding the sum.
Consider 23 + 45 = 68. Rewrite using the expanded form.
23 + 45 = (20 + 3) + (40 + 5). Rearrange.
(3 + 5) + (20 + 40). And then do the addition.
8 + 60 Add again.
68.
Notice that this closely matches the actual calculation you would do for 23 + 45. The 3 and 5 are added first, then the 2 and 4 are added in the tens place. We don't normally think of all the steps involved.

The Associative Property of Addition says that the order we do additions does not matter for the final result.

Associative Property of Addition
$(a + b) + c = a + (b + c)$

For example, (2 + 3) + 4 = 2 + (3 + 4).
When we add on the left side we get 5 + 4 = 9, and when we add on the right side we get 2 + 7 = 9.
Either gets the same final result even though the intermediate results are different.

The Associative Property of Addition combined with the Commutative Property of Addition allows us to rearrange the order of the addends and how they are grouped. We do that when we add numbers with more than one digit all of the time! Here we are just recognizing and naming properties that allow us to do this.

Note:
Exercise:

Problem:

Mark is training for a bicycle race. Last week he rode 18 miles on Monday, 15 miles on Wednesday, 26 miles on Friday, 49 miles on Saturday, and 32 miles on Sunday. What is the total number of miles he rode last week?

Solution:

He rode 140 miles.

Note:
Exercise:

Problem:

Lincoln Middle School has three grades. The number of students in each grade is 230, 165, and 325. What is the total number of students?

Solution:

The total number is 720 students.

Measuring Length

The average length of a newborn human is approximately 20 inches. The average length of a newborn giraffe is approximately 6 feet. Since 20 is greater than 6, does that mean that human newborns are taller than giraffe newborns?

We know the newborn giraffe is much taller, so what is the explanation? The units of measurement, inches and feet, are different so they should not be directly compared. Inches are much smaller than feet; it takes 12 inches to be the same length as 1 foot.

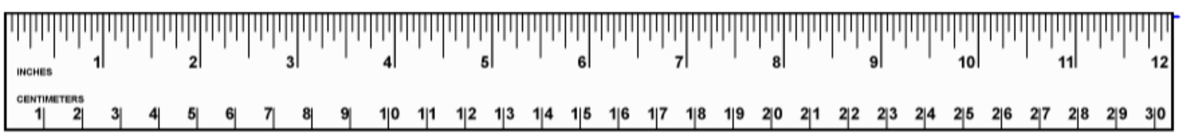
Tools such as rulers, yardsticks, tape measures are used to measure length or distance. We measure length along a path from a starting point to an ending point.

Most often, this is a straight path, but not always. For example, a tailor might use a flexible tape measure to find the distance around your waist or neck.

A ruler is very much like a number line. There is a mark for the starting point, 0, and then one unit to the right is “1”. Moving one unit further to the right is “2”, and this continues until the end of the ruler.

Often rulers have markings on them for two different units of measurement: inches and centimeters. Inches are bigger than centimeters, so for the same length the inches reading will be a smaller number than the centimeter reading. 30 centimeters is a little smaller than 12 inches while 31 centimeters is a little bigger than 12 inches. If a ruler is 12 inches long, the last centimeter marked on the ruler is 30 because 31 will not quite fit.

Note: This ruler is not to scale.

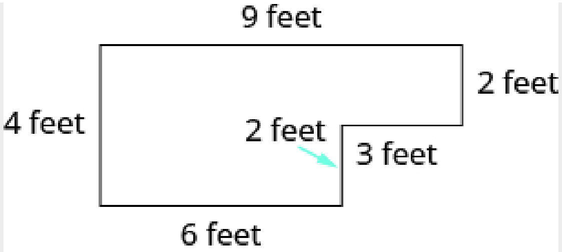


Video
For more information and help with finding the perimeter, please watch this video: https://www.youtube.com/watch?v=AAY1bsazcgM

Some application problems involve shapes. For example, a person might need to know the distance around a garden to put up a fence or around a picture to frame it. The **perimeter** is the distance around a geometric figure. The perimeter of a figure is the sum of the lengths of its sides.

Example:
Exercise:

Problem: Find the perimeter of the patio shown.

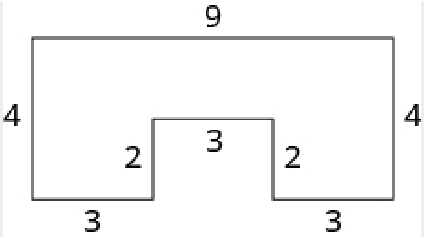


Solution:
Solution

We are asked to find the perimeter.	
Write a phrase.	the sum of the sides
Translate to math notation.	$4+6+2+3+2+9$
Simplify by adding.	26
Write a sentence to answer the question.	
We added feet, so the sum is 26 feet.	The perimeter of the patio is 26 feet.

Note:
Exercise:

Problem: Find the perimeter of each figure. All lengths are in inches.

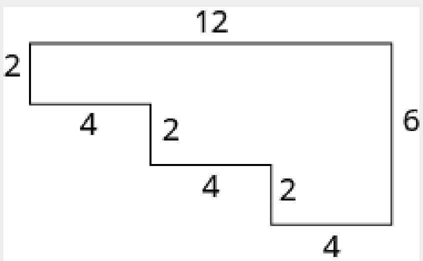


Solution:

The perimeter is 30 inches.

Note:
Exercise:

Problem: Find the perimeter of each figure. All lengths are in inches.



Solution:

The perimeter is 36 inches.

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Adding Two-Digit Numbers with base-10 blocks
- Adding Three-Digit Numbers with base-10 blocks
- Adding Whole Numbers

Key Concepts

- **Addition Notation** To describe addition, we can use symbols and words.

Operation	Notation	Expression	Read as	Result
Addition	$+$	$3+4$	three plus four	the sum of 3 and 4

- **Identity Property of Addition**
    - The sum of any number a and 0 is the number. $a+0=a$ $0+a=a$
- **Commutative Property of Addition**
    - Changing the order of the addends a and b does not change their sum. $a+b=b+a$.
- **Associative Property of Addition**
    - Changing the grouping of the addends a, b and c does not change their sum. $(a+b)+c=a+(b+c)$.
- **Add whole numbers.**

  Write the numbers so each place value lines up vertically.
  Add the digits in each place value. Work from right to left starting with the ones place. If a sum in a place value is more than 9, carry to the next place value.
  Continue adding each place value from right to left, adding each place value and carrying if needed.

Exercises

Practice Makes Perfect

Use Addition Notation

In the following exercises, translate the following from math expressions to words.

Exercise:

Problem: $5 + 2$

Solution:

five plus two; the sum of 5 and 2.

Exercise:

Problem: $6 + 3$

Exercise:

Problem: $13 + 18$

Solution:

thirteen plus eighteen; the sum of 13 and 18.

Exercise:

Problem: $15 + 16$

Exercise:

Problem: $214 + 642$

Solution:

two hundred fourteen plus six hundred forty-two; the sum of 214 and 642

Exercise:

Problem: $438 + 113$

Model Addition of Whole Numbers

In the following exercises, model the addition.

Exercise:

Problem: $2 + 4$

Solution:

$2 + 4 = 6$

Exercise:

Problem: $5 + 3$

Exercise:

Problem: $8 + 4$

Solution:

$8 + 4 = 12$

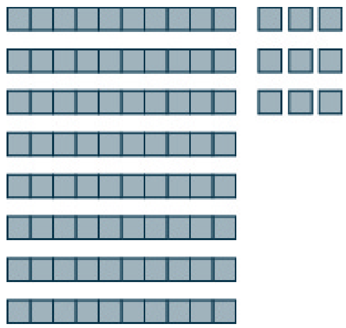
Exercise:

Problem: $5 + 9$

Exercise:

Problem: $14 + 75$

Solution:



$14+75=89$

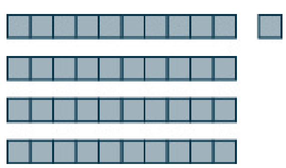
Exercise:

Problem: $15+63$

Exercise:

Problem: $16+25$

Solution:



$16+25=41$

Exercise:

Problem: $14+27$

Add Whole Numbers

In the following exercises, fill in the missing values in each chart.

Exercise:

Problem:

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2		4	5	6	7		9
1	1	2	3	4			7	8	9	
2		3	4	5	6		8			11
3	3		5		7	8		10		12
4	4	5			8	9		11	12	
5	5	6	7	8			11		13	
6	6	7	8		10			13		15
7			9	10		12			15	16
8	8	9		11			14		16	
9	9	10	11		13	14			17	

Solution:

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Exercise:

Problem:

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4		6		8	9
1	1	2	3		5	6		8		10
2	2		4		6	7		9	10	
3		4		6			9		11	
4	4	5	6	7			10	11		13
5	5	6		8	9		11	12	13	
6			8	9			12	13		15
7	7	8		10		12			15	16
8	8	9	10		12		14		16	17
9			11	12	13			16		

Exercise:

Problem:

+	3	4	5	6	7	8	9
6							
7							
8							
9							

Solution:

+	3	4	5	6	7	8	9
6	9	10	11	12	13	14	15
7	10	11	12	13	14	15	16
8	11	12	13	14	15	16	17
9	12	13	14	15	16	17	18

Exercise:

Problem:

+	6	7	8	9
3				
4				
5				
6				
7				
8				
9				

Exercise:

Problem:

+	5	6	7	8	9
5					
6					
7					
8					
9					

Solution:

+	5	6	7	8	9
5	10	11	12	13	14
6	11	12	13	14	15
7	12	13	14	15	16
8	13	14	15	16	17
9	14	15	16	17	18

Exercise:

Problem:

+	6	7	8	9
6				
7				
8				
9				

In the following exercises, add.

Exercise:

Problem:

ⓐ $0 + 13$
ⓑ $13 + 0$

Solution:

ⓐ 13
ⓑ 13

Exercise:

Problem:

ⓐ $0 + 5{,}280$
ⓑ $5{,}280 + 0$

Exercise:

Problem:

ⓐ $8 + 3$
ⓑ $3 + 8$

Solution:

ⓐ 11
ⓑ 11

Exercise:

Problem:

ⓐ $7 + 5$
ⓑ $5 + 7$

Exercise:

Problem: $45 + 33$

Solution:

78

Exercise:

Problem: $37 + 22$

Exercise:

Problem: $71 + 28$

Solution:

99

Exercise:

Problem: $43 + 53$

Exercise:

Problem: $26 + 59$

Solution:

85

Exercise:

Problem: $38 + 17$

Exercise:

Problem: $64 + 78$

Solution:

142

Exercise:

Problem: $92 + 39$

Exercise:

Problem: $168 + 325$

Solution:

493

Exercise:

Problem: $247 + 149$

Exercise:

Problem: $584 + 277$

Solution:

861

Exercise:

Problem: $175 + 648$

Exercise:

Problem: $832 + 199$

Solution:

$1,031$

Exercise:

Problem: $775 + 369$

Exercise:

Problem: $6,358 + 492$

Solution:

$6,850$

Exercise:

Problem: $9,184 + 578$

Exercise:

Problem: $3,740 + 18,593$

Solution:

$22,333$

Exercise:

Problem: $6{,}118 + 15{,}990$

Exercise:

Problem: $485{,}012 + 619{,}848$

Solution:

$1{,}104{,}860$

Exercise:

Problem: $368{,}911 + 857{,}289$

Exercise:

Problem: $24{,}731 + 592 + 3{,}868$

Solution:

$29{,}191$

Exercise:

Problem: $28{,}925 + 817 + 4{,}593$

Exercise:

Problem: $8{,}015 + 76{,}946 + 16{,}570$

Solution:

$101{,}531$

Exercise:

Problem: $6{,}291 + 54{,}107 + 28{,}635$

Translate Word Phrases to Math Notation

In the following exercises, translate each phrase into math notation and then simplify.

Exercise:

Problem: the sum of 13 and 18

Solution:

$13+18=31$

Exercise:

Problem: the sum of 12 and 19

Exercise:

Problem: the sum of 90 and 65

Solution:

$90+65=155$

Exercise:

Problem: the sum of 70 and 38

Exercise:

Problem: 33 increased by 49

Solution:

$33+49=82$

Exercise:

Problem: 68 increased by 25

Exercise:

Problem: 250 more than 599

Solution:

$250 + 599 = 849$

Exercise:

Problem: 115 more than 286

Exercise:

Problem: the total of 628 and 77

Solution:

$628 + 77 = 705$

Exercise:

Problem: the total of 593 and 79

Exercise:

Problem: 1,482 added to 915

Solution:

$915 + 1{,}482 = 2{,}397$

Exercise:

Problem: 2,719 added to 682

Add Whole Numbers in Applications

In the following exercises, solve the problem.

Exercise:

Problem:

Home remodeling Sophia remodeled her kitchen and bought a new range, microwave, and dishwasher. The range cost $1,100, the microwave cost $250, and the dishwasher cost $525. What was the total cost of these three appliances?

Solution:

The total cost was $1,875.

Exercise:

Problem:

Sports equipment Aiden bought a baseball bat, helmet, and glove. The bat cost $299, the helmet cost $35, and the glove cost $68. What was the total cost of Aiden's sports equipment?

Exercise:

Problem:

Bike riding Ethan rode his bike 14 miles on Monday, 19 miles on Tuesday, 12 miles on Wednesday, 25 miles on Friday, and 68 miles on Saturday. What was the total number of miles Ethan rode?

Solution:

Ethan rode 138 miles.

Exercise:

Problem:

Business Chloe has a flower shop. Last week she made 19 floral arrangements on Monday, 12 on Tuesday, 23 on Wednesday, 29 on Thursday, and 44 on Friday. What was the total number of floral arrangements Chloe made?

Exercise:

Problem:

Apartment size Jackson lives in a 7 room apartment. The number of square feet in each room is 238, 120, 156, 196, 100, 132, and 225. What is the total number of square feet in all 7 rooms?

Solution:

The total square footage in the rooms is 1,167 square feet.

Exercise:

Problem:

Weight Seven men rented a fishing boat. The weights of the men were $175, 192, 148, 169, 205, 181,$ and 225 pounds. What was the total weight of the seven men?

Exercise:

Problem:

Salary Last year Natalie's salary was $\$82,572$. Two years ago, her salary was $\$79,316$, and three years ago it was $\$75,298$. What is the total amount of Natalie's salary for the past three years?

Solution:

Natalie's total salary is $237,186.

Exercise:

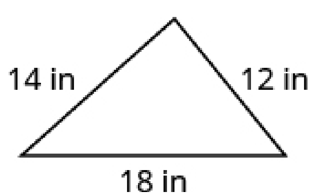
Problem:

Home sales Emma is a realtor. Last month, she sold three houses. The selling prices of the houses were $\$292,540$, $\$505,875$, and $\$423,699$. What was the total of the three selling prices?

In the following exercises, find the perimeter of each figure.

Exercise:

Problem:

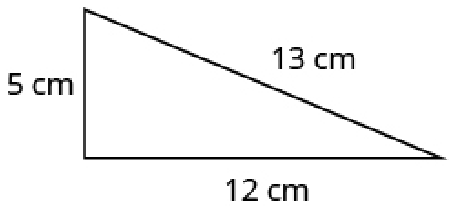


Solution:

The perimeter of the figure is 44 inches.

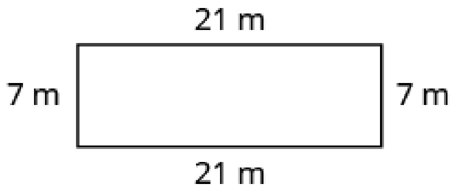
Exercise:

Problem:



Exercise:

Problem:

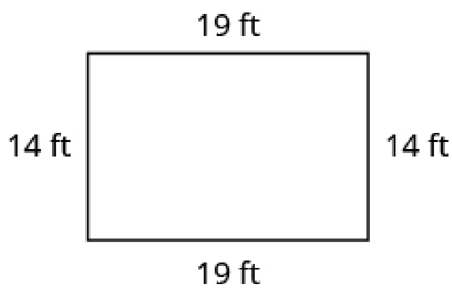


Solution:

The perimeter of the figure is 56 meters.

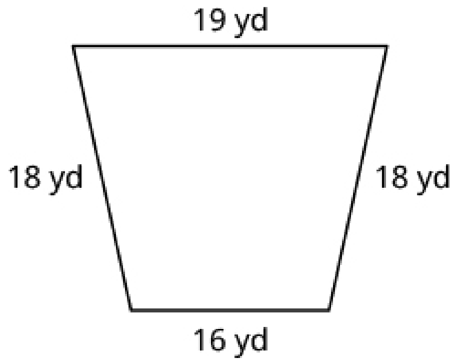
Exercise:

Problem:



Exercise:

Problem:

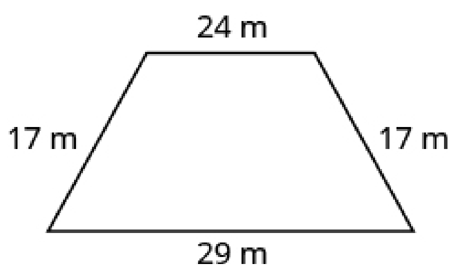


Solution:

The perimeter of the figure is 71 yards.

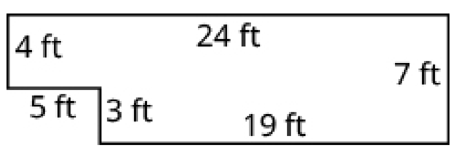
Exercise:

Problem:



Exercise:

Problem:

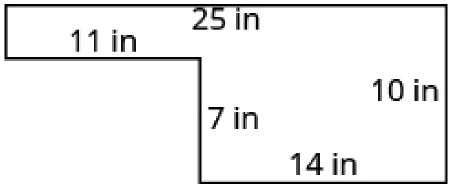


Solution:

The perimeter of the figure is 62 feet.

Exercise:

Problem:



Everyday Math

Exercise:

Problem:

Calories Paulette had a grilled chicken salad, ranch dressing, and a 16-ounce drink for lunch. On the restaurant's nutrition chart, she saw that each item had the following number of calories:

Grilled chicken salad – 320 calories
Ranch dressing – 170 calories
16-ounce drink – 150 calories

What was the total number of calories of Paulette's lunch?

Solution:

The total number of calories was 640.

Exercise:

Problem:

Calories Fred had a grilled chicken sandwich, a small order of fries, and a 12-oz chocolate shake for dinner. The restaurant's nutrition chart lists the following calories for each item:

Grilled chicken sandwich – 420 calories
Small fries – 230 calories
12-oz chocolate shake – 580 calories

What was the total number of calories of Fred's dinner?

Exercise:

Problem:

Test scores A students needs a total of 400 points on five tests to pass a course. The student scored $82, 91, 75, 88, \text{and } 70$. Did the student pass the course?

Solution:

Yes, he scored 406 points.

Exercise:

Problem:

Elevators The maximum weight capacity of an elevator is 1150 pounds. Six men are in the elevator. Their weights are $210, 145, 183, 230, 159, \text{and } 164$ pounds. Is the total weight below the elevators' maximum capacity?

Writing Exercises

Exercise:

Problem:

How confident do you feel about your knowledge of the addition facts? If you are not fully confident, what will you do to improve your skills?

Solution:

Answers will vary.

Exercise:

Problem: How have you used models to help you learn the addition facts?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
use addition notation.			
model addition of whole numbers.			
add whole numbers without models.			
explain when and how addition can be more efficient than counting.			
recognize and explain the commutative and associative properties of addition.			
translate word phrases to math notation.			
add whole numbers in applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

addend
: A number that is added.

expression
: A math statement that includes numbers and operations.

perimeter
: The perimeter is the distance around a geometric figure.

sum
: The sum is the result of adding two or more numbers.

variable
: A letter or symbol used to stand for potentially any value.

Subtract Whole Numbers
By the end of this section, you will be able to:

- Use subtraction notation
- Model subtraction of whole numbers
- Subtract whole numbers
- Identify properties of subtraction
- Translate word phrases to math notation
- Subtract whole numbers in applications

Note:
Before you get started, take this readiness quiz.

1. Model $3+4$ using base-ten blocks.
   If you missed this problem, review [link].
2. Add: $324+586$.
   If you missed this problem, review [link].

Use Subtraction Notation

Suppose there are seven bananas in a bowl. Elana uses three of them to make a smoothie. How many bananas are left in the bowl? To answer the question, we subtract three from seven. When we subtract, we take one number away from another to find the **difference**. The notation we use to subtract 3 from 7 is

Equation:

$$7-3$$

We read $7-3$ as *seven minus three* and the result is *the difference of seven and three*.

Note:
Subtraction Notation
To describe subtraction, we can use symbols and words.

Operation	**Notation**	**Expression**	**Read as**	**Result**
Subtraction	$-$	$7-3$	seven minus three	the difference of 7 and 3

Example:
Exercise:

Problem:

Translate from math notation to words: ⓐ $8-1$ ⓑ $26-14$.

Solution:
Solution

- ⓐ We read this as *eight minus one*. The result is *the difference of eight and one*.
- ⓑ We read this as *twenty-six minus fourteen*. The resuilt is *the difference of twenty-six and fourteen*.

Note:
Translate from math notation to words:
Exercise:

Problem:

ⓐ $12 - 4$
ⓑ $29 - 11$

Solution:

ⓐ twelve minus four; the difference of twelve and four
ⓑ twenty-nine minus eleven; the difference of twenty-nine and eleven

Note:
Translate from math notation to words:
Exercise:

Problem:

ⓐ $11 - 2$
ⓑ $29 - 12$

Solution:

ⓐ eleven minus two; the difference of eleven and two
ⓑ twenty-nine minus twelve; the difference of twenty-nine and twelve

Model Subtraction of Whole Numbers

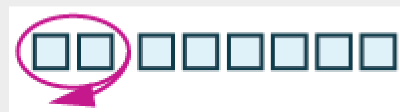
A model can help us visualize the process of subtraction much as it did with addition. Again, we will use base-10 blocks. Remember a block represents 1 and a rod represents 10. Let’s start by modeling the subtraction expression we just considered, $7 - 3$.

We start by modeling the first number, 7.	
Now take away the second number, 3. We'll circle 3 blocks to show that we are taking them away.	
Count the number of blocks remaining.	
There are 4 ones blocks left.	We have shown that $\mathbf{7 - 3 = 4}$.

Example:
Exercise:

Problem: Model the subtraction: $8 - 2$.

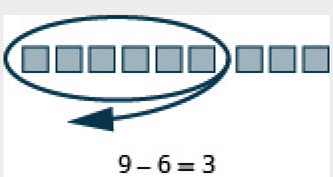
Solution:
Solution

$8-2$ means the difference of 8 and 2.	
Model the first, 8.	
Take away the second number, 2.	
Count the number of blocks remaining.	
There are 6 ones blocks left.	We have shown that $8-2=6$.

Note:
Exercise:

Problem: Model: $9-6$.

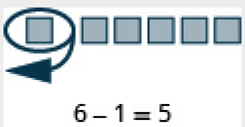
Solution:



Note:
Exercise:

Problem: Model: $6 - 1$.

Solution:

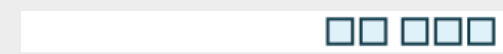


Example:
Exercise:

Problem: Model the subtraction: $13 - 8$.

Solution:
Solution

Model the first number, 13. We use 1 ten and 3 ones.	
Take away the second number, 8. However, there are not 8 ones, so we will exchange	

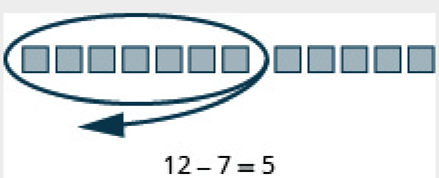
the 1 ten for 10 ones.	
Now we can take away 8 ones.	
Count the blocks remaining.	
There are five ones left.	We have shown that $13 - 8 = 5$.

As we did with addition, we can describe the models as ones blocks and tens rods, or we can simply say ones and tens.

Note:
Exercise:

Problem: Model the subtraction: $12 - 7$.

Solution:

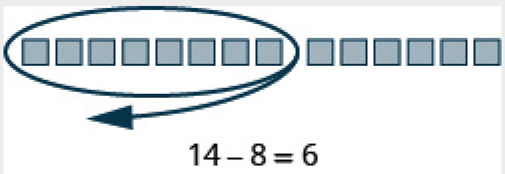


Note:

Exercise:

Problem: Model the subtraction: 14 − 8.

Solution:

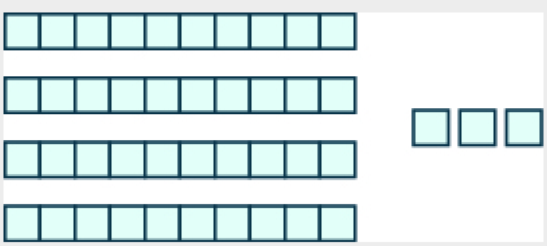


Example:
Exercise:

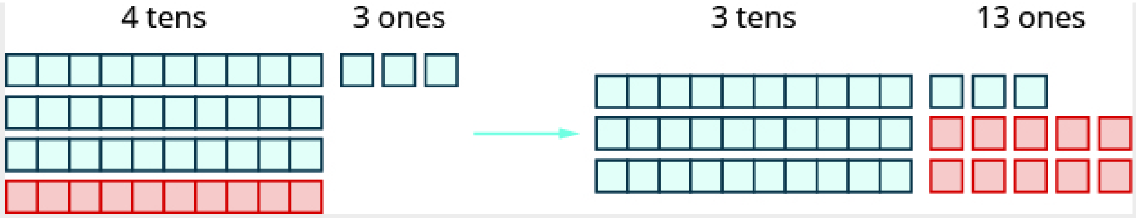
Problem: Model the subtraction: 43 − 26.

Solution:
Solution

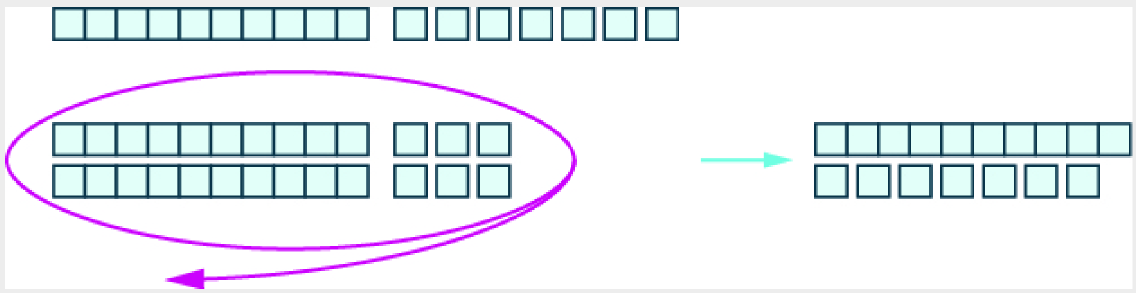
Because 43 − 26 means 43 take away 26, we begin by modeling the 43.



Now, we need to take away 26, which is 2 tens and 6 ones. We cannot take away 6 ones from 3 ones. So, we exchange 1 ten for 10 ones.



Now we can take away 2 tens and 6 ones.



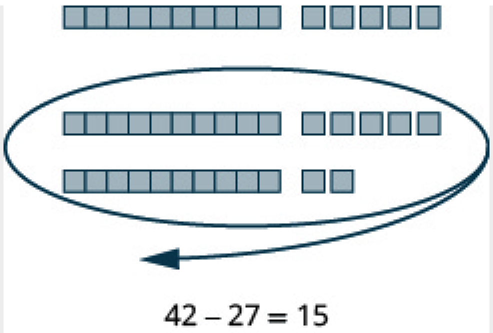
Count the number of blocks remaining. There is 1 ten and 7 ones, which is 17.

$43 - 26 = 17$

Note:
Exercise:

Problem: Model the subtraction: $42 - 27$.

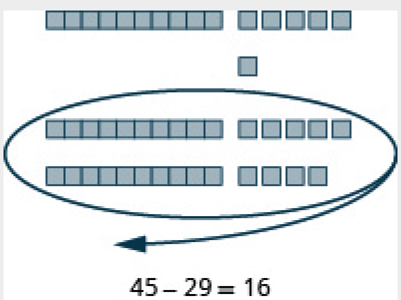
Solution:



Note:
Exercise:

Problem: Model the subtraction: $45 - 29$.

Solution:



Properties of Subtraction

Just as addition has some interesting properties (identity, commutative, and associative) subtraction has interesting properties too. Since addition and subtraction are related, we might expect there to be similar properties for subtraction, but we need to investigate to be sure.

Identity Property of Subtraction
The identity property for addition is: $a + 0 = a$. A similar property for subtraction is: $a - 0 = a$. Examples are: $5 - 0 = 5$ and $17 - 0 = 17$.

No matter how many examples we give it does not prove the property is true for any value of a. In order for us to accept this as a property it must be true for every value of a. That requires more general reasoning. If I have an amount a and remove nothing from it then my amount is still a.

Here is another property related to the identity property of subtraction: $a - a = 0$. In words this says, any number minus the same number is zero. Examples are $5 - 5 = 0$ and $17 - 17 = 0$. Just giving examples does not prove that this is true in general. In your own words, explain why this is true.

Commutative Property of Subtraction?
The commutative property of addition says that $a + b = b + a$. The order of the addends does not matter. What about for subtraction. Does $a - b = b - a$?

Sometimes it helps to put numbers in for the variables to see if a property works. If it works for those numbers the property might be true but that doesn't prove that it is true with other values. But if it is false then we know that our property isn't a real property. It takes just one counterexample to show the property is false.

Let $a = 7$ and $b = 3$. Then $a - b$ is $7 - 3 = 4$, while $b - a$ is $3 - 7$ which we can not compute with whole numbers because one can not take a larger amount away from a smaller amount. Therefore there is no commutative property of subtraction.

Associative Property of Subtraction?
The associative property of addition says that $(a + b) + c = a + (b + c)$.
If there is an associative property of subtraction it would look the same except every plus sign would be replaced with a minus sign: $(a - b) - c = a - (b - c)$.

This also does not work. Try $a = 5$, $b = 3$, and $c = 1$ to show that it is not true.

(5 - 3) - 1 = 2 - 1 = 1 while 5 - (3 - 1) = 5 - 2 = 3. Therefore there is no associative property of subtraction.

Subtract Whole Numbers

Addition and subtraction are inverse operations. Addition undoes subtraction, and subtraction undoes addition.

If you have $7 and loan $3 then you have $4 left. If you then get the $3 paid back you have $7 again.
7 - 3 + 3 = 4 + 3 = 7
Subtracting 3 was undone by adding 3.

It works the other way too.
If you have $4 and find $3 and then lose $3 you'll be back where you started with $4.
4 + 3 - 3 = 7 - 3 = 4.
Adding 3 was undone by subtracting 3.

One way to know that $7-3=4$ is because $4+3=7$. Knowing all the addition number facts will help with subtraction. We can check subtraction by adding.

Equation:

$$\begin{array}{ccc} 7-3=4 & \text{because} & 4+3=7 \\ 13-8=5 & \text{because} & 5+8=13 \\ 43-26=17 & \text{because} & 17+26=43 \end{array}$$

Example:
Exercise:

Problem: Subtract and then check by adding:

ⓐ $9-7$

ⓑ $8-3$.

Solution:
Solution

ⓐ	
	$9-7$
Subtract 7 from 9.	2
Check with addition. $2+7=9\checkmark$	

ⓑ	
	$8-3$
Subtract 3 from 8.	5
Check with addition. $5+3=8\checkmark$	

Note:
Exercise:

Problem: Subtract and then check by adding:

$7 - 0$

Solution:

$7 - 0 = 7; 7 + 0 = 7$

Note:
Exercise:

Problem: Subtract and then check by adding:

$6 - 2$

Solution:

$6 - 2 = 4; 2 + 4 = 6$

To subtract numbers with more than one digit, it is usually easier to write the numbers vertically in columns just as we did for addition. Align the digits by place value, and then subtract each column starting with the ones and then working to the left.

Example:
Exercise:

Problem: Subtract and then check by adding: $89 - 61$.

Solution:
Solution

Write the numbers so the ones and tens digits line up vertically.	$\begin{array}{r} 89 \\ -61 \\ \hline \end{array}$
Subtract the digits in each place value. Subtract the ones: $9 - 1 = 8$ Subtract the tens: $8 - 6 = 2$	$\begin{array}{r} 89 \\ -61 \\ \hline 28 \end{array}$
Check using addition. $\begin{array}{r} 28 \\ +61 \\ \hline 89 \end{array}$	

Our answer is correct.

Note:
Exercise:

Problem: Subtract and then check by adding: $86 - 54$.

Solution:

$86 - 54 = 32$ because $54 + 32 = 86$

Note:
Exercise:

Problem: Subtract and then check by adding: $99 - 74$.

Solution:

$99 - 74 = 25$ because $74 + 25 = 99$

When we modeled subtracting 26 from 43, we exchanged 1 ten for 10 ones. When we do this without the model, we say we borrow 1 from the tens place and add 10 to the ones place.

Note:
Find the difference of whole numbers.

Write the numbers so each place value lines up vertically.
Subtract the digits in each place value. Work from right to left starting with the ones place. If the digit on top is less than the digit below, borrow as needed.
Continue subtracting each place value from right to left, borrowing if needed.
Check by adding.

Why do we start with the digits on the right instead of the digits on the left?
If we started on the left and wrote down the answer we would have to go back and correct it if we needed to borrow when we got further to the right.

Example:
Exercise:

Problem: Subtract: $43 - 26$.

Solution:
Solution

Write the numbers so each place value lines up vertically.	
Subtract the ones. We cannot subtract 6 from 3, so we borrow 1 ten. This makes 3 tens and 13 ones. We write these numbers above each place and cross out the original digits.	
Now we can subtract the ones. $13 - 6 = 7$. We write the 7 in the ones place in the difference.	
Now we subtract the tens. $3 - 2 = 1$. We write the 1 in the tens place in the difference.	
Check by adding. $\begin{array}{r} 17 \\ +26 \\ \hline 43 \checkmark \end{array}$ Our answer is correct.	

Note:
Exercise:

Problem: Subtract and then check by adding: $93 - 58$.

Solution:

$93 - 58 = 35$ because $58 + 35 = 93$

Note:
Exercise:

Problem: Subtract and then check by adding: $81 - 39$.

Solution:

$81 - 39 = 42$ because $42 + 39 = 81$

Example:
Exercise:

Problem: Subtract and then check by adding: $207 - 64$.

Solution:
Solution

Write the numbers so each place value lines up vertically.	
Subtract the ones. $7 - 4 = 3$. Write the 3 in the ones place in the difference. Write the 3 in the ones place in the difference.	
Subtract the tens. We cannot subtract 6 from 0 so we borrow 1 hundred and add 10 tens to the 0 tens we had. This makes a total of 10 tens. We write 10 above the tens place and cross out the 0. Then we cross out the 2 in the hundreds place and write 1 above it.	
Now we subtract the tens. $10 - 6 = 4$. We write the 4 in the tens place in the difference.	
Finally, subtract the hundreds. There is no digit in the hundreds place in the bottom number so we can imagine a 0 in that place. Since $1 - 0 = 1$, we write 1 in the hundreds place in the difference.	
Check by adding. $\begin{array}{r} \overset{1}{1}43 \\ +\ 64 \\ \hline 207 \end{array}$ ✓ Our answer is correct.	

Note:
Exercise:

Problem: Subtract and then check by adding: $439 - 52$.

Solution:

$439 - 52 = 387$ because $387 + 52 = 439$

Note:
Exercise:

Problem: Subtract and then check by adding: $318 - 75$.

Solution:

$318 - 75 = 243$ because $243 + 75 = 318$

Example:
Exercise:

Problem: Subtract and then check by adding: $910 - 586$.

Solution:
Solution

Write the numbers so each place value lines up vertically.	

Subtract the ones. We cannot subtract 6 from 0, so we borrow 1 ten and add 10 ones to the 10 ones we had. This makes 10 ones. We write a 0 above the tens place and cross out the 1. We write the 10 above the ones place and cross out the 0. Now we can subtract the ones. $10 - 6 = 4$.

Write the 4 in the ones place of the difference.

Subtract the tens. We cannot subtract 8 from 0, so we borrow 1 hundred and add 10 tens to the 0 tens we had, which gives us 10 tens. Write 8 above the hundreds place and cross out the 9. Write 10 above the tens place.

Now we can subtract the tens. $10 - 8 = 2$.

Subtract the hundreds place. $8 - 5 = 3$ Write the 3 in the hundreds place in the difference.

Check by adding.

$$\begin{array}{r} \overset{1}{3}\overset{1}{2}4 \\ +\ 586 \\ \hline 910 \end{array} \checkmark$$

Our answer is correct.

Note:
Exercise:

Problem: Subtract and then check by adding: $832 - 376$.

Solution:

832 − 376 = 456 because 456 + 376 = 832

Note:
Exercise:

Problem: Subtract and then check by adding: $847 - 578$.

Solution:

847 − 578 = 269 because 269 + 578 = 847

Example:
Exercise:

Problem: Subtract and then check by adding: $2{,}162 - 479$.

Solution:
Solution

Write the numbers so each place values line up vertically.	$\begin{array}{r} 2,162 \\ -\ 479 \\ \hline \end{array}$

Subtract the ones. Since we cannot subtract 9 from 2, borrow 1 ten and add 10 ones to the 2 ones to make 12 ones. Write 5 above the tens place and cross out the 6. Write 12 above the ones place and cross out the 2.	$\begin{array}{r} {\scriptstyle 5\ 12} \\ 2,1\cancel{6}\cancel{2} \\ -\ 479 \\ \hline \end{array}$
Now we can subtract the ones.	$12 - 9 = 3$
Write 3 in the ones place in the difference.	$\begin{array}{r} {\scriptstyle 5\ 12} \\ 2,1\cancel{6}\cancel{2} \\ -\ 479 \\ \hline 3 \end{array}$
Subtract the tens. Since we cannot subtract 7 from 5, borrow 1 hundred and add 10 tens to the 5 tens to make 15 tens. Write 0 above the hundreds place and cross out the 1. Write 15 above the tens place.	$\begin{array}{r} {\scriptstyle 15\ \ \ } \\ {\scriptstyle 0\ \cancel{5}\ 12} \\ 2,\cancel{1}\cancel{6}\cancel{2} \\ -\ 479 \\ \hline 3 \end{array}$
Now we can subtract the tens.	$15 - 7 = 8$
Write 8 in the tens place in the difference.	$\begin{array}{r} {\scriptstyle 0\ 15\ 12} \\ 2,\cancel{1}\cancel{6}\cancel{2} \\ -\ 479 \\ \hline 83 \end{array}$
Now we can subtract the hundreds.	$\begin{array}{r} {\scriptstyle 10\ \ \ \ \ \ \ } \\ {\scriptstyle 1\ \ \cancel{0}\ 15\ 12} \\ \cancel{2},\cancel{1}\cancel{6}\cancel{2} \\ -\ 479 \\ \hline 83 \end{array}$

Write 6 in the hundreds place in the difference.	$\begin{array}{r} {\scriptstyle 1}\ \ {\scriptstyle 10}\,{\scriptstyle 15}\,{\scriptstyle 12} \\ \cancel{2},\cancel{1}\,\cancel{6}\,\cancel{2} \\ -\ \ 4\ 7\ 9 \\ \hline 6\ 8\ 3 \end{array}$
Subtract the thousands. There is no digit in the thousands place of the bottom number, so we imagine a 0. $1 - 0 = 1$. Write 1 in the thousands place of the difference.	$\begin{array}{r} {\scriptstyle 1}\ \ {\scriptstyle 10}\,\cancel{\scriptstyle 15}\,\cancel{\scriptstyle 12} \\ \cancel{2},\cancel{1}\,\cancel{6}\,\cancel{2} \\ -\ \ 4\ 7\ 9 \\ \hline 1,6\ 8\ 3 \end{array}$
Check by adding. $\begin{array}{r} {\scriptstyle 1}\ \ \ {\scriptstyle 1}\,{\scriptstyle 1}\ \ \\ 1,683 \\ +\ \ 479 \\ \hline 2,162 \checkmark \end{array}$	

Our answer is correct.

Note:
Exercise:

Problem: Subtract and then check by adding: $4{,}585 - 697$.

Solution:

4,585 − 697 = 3,888 because 3,888 + 697 = 4,585

Note:
Exercise:

Problem: Subtract and then check by adding: $5,637 - 899$.

Solution:

5,637 − 899 = 4,738 because 4,738 + 899 = 5,637

Translate Word Phrases to Math Notation

As with addition, word phrases can tell us to operate on two numbers using subtraction. To translate from a word phrase to math notation, we look for key words that indicate subtraction. Some of the words that indicate subtraction are listed in [link].

Operation	Word Phrase	Example	Expression
Subtraction	minus	5 minus 1	$5 - 1$
	difference	the difference of 9 and 4	$9 - 4$
	decreased by	7 decreased by 3	$7 - 3$
	less than	5 less than 8	$8 - 5$

Operation	Word Phrase	Example	Expression
	subtracted from	1 subtracted from 6	$6 - 1$

Example:
Exercise:

Problem: Translate and then simplify:

ⓐ the difference of 13 and 8
ⓑ subtract 24 from 43

Solution:
Solution

- ⓐ

  The word *difference* tells us to subtract the two numbers. The numbers stay in the same order as in the phrase.

	the difference of 13 and 8
Translate.	$13 - 8$
Simplify.	5

- ⓑ

  The words *subtract from* tells us to take the second number away from the first. We must be careful to get the order correct.

	subtract 24 from 43
Translate.	$43 - 24$
Simplify.	19

Note:
Exercise:

Problem: Translate and simplify:

ⓐ the difference of 14 and 9
ⓑ subtract 21 from 37

Solution:

ⓐ 14 − 9 = 5
ⓑ 37 − 21 = 16

Note:

Exercise:

Problem: Translate and simplify:

ⓐ 11 decreased by 6
ⓑ 18 less than 67

Solution:

ⓐ $11 - 6 = 5$
ⓑ $67 - 18 = 49$

Subtract Whole Numbers in Applications

To solve applications with subtraction, we will use the same plan that we used with addition.

- Determine what we are asked to find.
- Write a phrase that gives the information to find it.
- Translate the phrase into math notation.
- Simplify to get the answer.
- Write a sentence to answer the question, using the appropriate units.

Example:
Exercise:

Problem:

The temperature in Chicago one morning was 73 degrees Fahrenheit. A cold front arrived and by noon the temperature was 27 degrees Fahrenheit. What was the difference between the temperature in the morning and the temperature at noon?

Solution:
Solution

We are asked to find the difference between the morning temperature and the noon temperature.

Write a phrase.	the difference of 73 and 27
Translate to math notation. *Difference* tells us to subtract.	$73 - 27$
Then we do the subtraction.	$\begin{array}{r} \overset{6}{\cancel{7}}\,\overset{13}{\cancel{3}} \\ -\ 2\,7 \\ \hline 4\,6 \end{array}$
Write a sentence to answer the question.	The difference in temperatures was 46 degrees Fahrenheit.

Note:
Exercise:

Problem:

The high temperature on June 1st in Boston was 77 degrees Fahrenheit, and the low temperature was 58 degrees Fahrenheit. What was the difference between the high and low temperatures?

Solution:

The difference is 19 degrees Fahrenheit.

Note:
Exercise:

Problem:

The weather forecast for June 2 in St Louis predicts a high temperature of 90 degrees Fahrenheit and a low of 73 degrees Fahrenheit. What is the difference between the predicted high and low temperatures?

Solution:

The difference is 17 degrees Fahrenheit.

Example:
Exercise:

Problem:

A washing machine is on sale for \$399. Its regular price is \$588. What is the difference between the regular price and the sale price?

Solution:
Solution

We are asked to find the difference between the regular price and the sale price.

Write a phrase.	the difference between 588 and 399
Translate to math notation.	$588 - 399$
Subtract.	$\begin{array}{r} \overset{4}{\cancel{5}}\overset{17}{\cancel{8}}\overset{18}{\cancel{8}} \\ -\ 399 \\ \hline 189 \end{array}$
Write a sentence to answer the question.	The difference between the regular price and the sale price is \$189.

Note:
Exercise:

Problem:

A television set is on sale for \$499. Its regular price is \$648. What is the difference between the regular price and the sale price?

Solution:

The difference is \$149.

Note:
Exercise:

Problem:

A patio set is on sale for $149. Its regular price is $285. What is the difference between the regular price and the sale price?

Solution:

The difference is $136.

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Model subtraction of two-digit whole numbers
- Model subtraction of three-digit whole numbers
- Subtract Whole Numbers

Key Concepts

Operation	Notation	Expression	Read as	Result

Operation	Notation	Expression	Read as	Result
Subtraction	$-$	$7-3$	seven minus three	the difference of 7 and 3

- **Subtract whole numbers.**

  Write the numbers so each place value lines up vertically.
  Subtract the digits in each place value. Work from right to left starting with the ones place. If the digit on top is less than the digit below, borrow as needed.
  Continue subtracting each place value from right to left, borrowing if needed.
  Check by adding.

Exercises

Practice Makes Perfect

Use Subtraction Notation

In the following exercises, translate from math notation to words.
Exercise:

Problem: $15-9$

Solution:

fifteen minus nine; the difference of fifteen and nine

Exercise:

Problem: $18-16$

Exercise:

Problem: $42 - 35$

Solution:

forty-two minus thirty-five; the difference of forty-two and thirty-five

Exercise:

Problem: $83 - 64$

Exercise:

Problem: $675 - 350$

Solution:

hundred seventy-five minus three hundred fifty; the difference of six hundred seventy-five and three hundred fifty

Exercise:

Problem: $790 - 525$

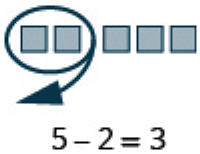
Model Subtraction of Whole Numbers

In the following exercises, model the subtraction.

Exercise:

Problem: $5 - 2$

Solution:



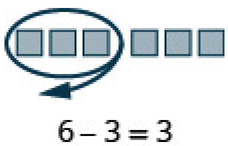
Exercise:

Problem: $8 - 4$

Exercise:

Problem: $6 - 3$

Solution:



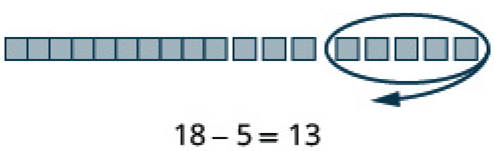
Exercise:

Problem: $7 - 5$

Exercise:

Problem: $18 - 5$

Solution:



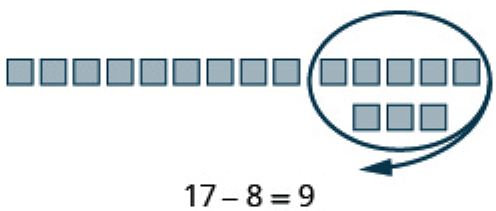
Exercise:

Problem: $19 - 8$

Exercise:

Problem: $17 - 8$

Solution:



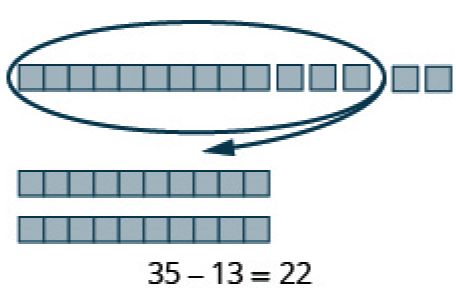
Exercise:

Problem: $17 - 9$

Exercise:

Problem: $35 - 13$

Solution:



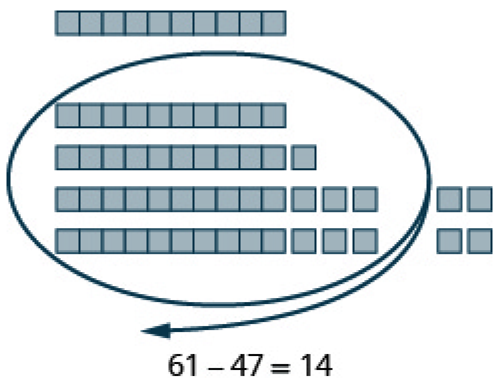
Exercise:

Problem: $32 - 11$

Exercise:

Problem: $61 - 47$

Solution:



Exercise:

Problem: $55 - 36$

Subtract Whole Numbers

In the following exercises, subtract and then check by adding.

Exercise:

Problem: $9 - 4$

Solution:

5

Exercise:

Problem: $9 - 3$

Exercise:

Problem: $8 - 0$

Solution:

8

Exercise:

Problem: $2 - 0$

Exercise:

Problem: $38 - 16$

Solution:

22

Exercise:

Problem: $45 - 21$

Exercise:

Problem: $85 - 52$

Solution:

33

Exercise:

Problem: $99 - 47$

Exercise:

Problem: $493 - 370$

Solution:

123

Exercise:

Problem: $268 - 106$

Exercise:

Problem: $5{,}946 - 4{,}625$

Solution:

$1{,}321$

Exercise:

Problem: $7{,}775 - 3{,}251$

Exercise:

Problem: $75 - 47$

Solution:

28

Exercise:

Problem: $63 - 59$

Exercise:

Problem: $461 - 239$

Solution:

222

Exercise:

Problem: $486 - 257$

Exercise:

Problem: $525 - 179$

Solution:

346

Exercise:

Problem: $542 - 288$

Exercise:

Problem: $6,318 - 2,799$

Solution:

3,519

Exercise:

Problem: $8,153 - 3,978$

Exercise:

Problem: $2,150 - 964$

Solution:

$1{,}186$

Exercise:

Problem: $4{,}245-899$

Exercise:

Problem: $43{,}650-8{,}982$

Solution:

$34{,}668$

Exercise:

Problem: $35{,}162-7{,}885$

Translate Word Phrases to Algebraic Expressions

In the following exercises, translate and simplify.

Exercise:

Problem: The difference of 10 and 3

Solution:

10 − 3; 7

Exercise:

Problem: The difference of 12 and 8

Exercise:

Problem: The difference of 15 and 4

Solution:

15 − 4; 11

Exercise:

Problem: The difference of 18 and 7

Exercise:

Problem: Subtract 6 from 9

Solution:

9 − 6; 3

Exercise:

Problem: Subtract 8 from 9

Exercise:

Problem: Subtract 28 from 75

Solution:

75 − 28; 47

Exercise:

Problem: Subtract 59 from 81

Exercise:

Problem: 45 decreased by 20

Solution:

45 − 20; 25

Exercise:

Problem: 37 decreased by 24

Exercise:

Problem: 92 decreased by 67

Solution:

92 − 67; 25

Exercise:

Problem: 75 decreased by 49

Exercise:

Problem: 12 less than 16

Solution:

16 − 12; 4

Exercise:

Problem: 15 less than 19

Exercise:

Problem: 38 less than 61

Solution:

61 − 38; 23

Exercise:

Problem: 47 less than 62

Mixed Practice

In the following exercises, simplify.

Exercise:

Problem: $76 - 47$

Solution:

29

Exercise:

Problem: $91 - 53$

Exercise:

Problem: $256 - 184$

Solution:

72

Exercise:

Problem: $305 - 262$

Exercise:

Problem: $719 + 341$

Solution:

1,060

Exercise:

Problem: $647 + 528$

Exercise:

Problem: $2{,}015 - 1{,}993$

Solution:

22

Exercise:

Problem: $2{,}020 - 1{,}984$

In the following exercises, translate and simplify.

Exercise:

Problem: Seventy-five more than thirty-five

Solution:

75 + 35; 110

Exercise:

Problem: Sixty more than ninety-three

Exercise:

Problem: 13 less than 41

Solution:

41 − 13; 28

Exercise:

Problem: 28 less than 36

Exercise:

Problem: The difference of 100 and 76

Solution:

$100 - 76$; 24

Exercise:

Problem: The difference of 1,000 and 945

Subtract Whole Numbers in Applications

In the following exercises, solve.

Exercise:

Problem:

Temperature The high temperature on June 2 in Las Vegas was 80 degrees and the low temperature was 63 degrees. What was the difference between the high and low temperatures?

Solution:

The difference between the high and low temperature was 17 degrees

Exercise:

Problem:

Temperature The high temperature on June 1 in Phoenix was 97 degrees and the low was 73 degrees. What was the difference between the high and low temperatures?

Exercise:

Problem:

Class size Olivia’s third grade class has 35 children. Last year, her second grade class had 22 children. What is the difference between the number of children in Olivia’s third grade class and her second grade class?

Solution:

The difference between the third grade and second grade was 13 children.

Exercise:

Problem:

Class size There are 82 students in the school band and 46 in the school orchestra. What is the difference between the number of students in the band and the orchestra?

Exercise:

Problem:

Shopping A mountain bike is on sale for $399. Its regular price is $650. What is the difference between the regular price and the sale price?

Solution:

The difference between the regular price and sale price is $251.

Exercise:

Problem:

Shopping A mattress set is on sale for $755. Its regular price is $1,600. What is the difference between the regular price and the sale price?

Exercise:

Problem:

Savings John wants to buy a laptop that costs $\$840$. He has $\$685$ in his savings account. How much more does he need to save in order to buy the laptop?

Solution:

John needs to save $155 more.

Exercise:

Problem:

Banking Mason had $\$1,125$ in his checking account. He spent $\$892$. How much money does he have left?

Everyday Math

Exercise:

Problem:

Road trip Noah was driving from Philadelphia to Cincinnati, a distance of 502 miles. He drove 115 miles, stopped for gas, and then drove another 230 miles before lunch. How many more miles did he have to travel?

Solution:

157 miles

Exercise:

Problem:

Test Scores Sara needs 350 points to pass her course. She scored $75, 50, 70,$ and 80 on her first four tests. How many more points does Sara need to pass the course?

Writing Exercises

Exercise:

Problem: Explain how subtraction and addition are related.

Solution:

Answers may vary.

Exercise:

Problem:

How does knowing addition facts help you to subtract numbers?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use subtraction notation.			
model subtraction of whole numbers.			
subtract whole numbers.			
identify properties of subtraction.			
translate word phrases to math notation.			
subtract whole numbers in applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

difference
: The difference is the result of subtracting two or more numbers.

Multiply Whole Numbers Intermediate Level
By the end of this section, you will be able to:

- Use multiplication notation
- Model multiplication of whole numbers
- Multiply whole numbers
- Translate word phrases to math notation
- Multiply whole numbers in applications

Note:
Before you get started, take this readiness quiz.

1. Add: $1,683+479$.
   If you missed this problem, review [link].
2. Subtract: $605-321$.
   If you missed this problem, review [link].

Use Multiplication Notation

Suppose you were asked to find out the number of pennies shown in [link].

Would you count the pennies individually? Or would you count the number of pennies in a row and add that number 3 times.
Equation:

$$8+8+8$$

Multiplication is a way to represent repeated addition. Repeated addition means the same addend occurs over and over again. So instead of adding three 8s, we could write a multiplication expression.
Equation:

$$3\times 8$$

We call each number in the multiplication a **factor** and the result the **product**. We read 3×8 as *three times eight*, and the result as *the product of three and eight*.

factor x factor = product

There are several symbols that represent multiplication. These include the symbol $\times$ as well as the dot, $\cdot$, and parentheses $(\ )$.

Note:
Operation Symbols for Multiplication
To describe multiplication, we can use symbols and words.

Operation	**Notation**	**Expression**	**Read as**	**Result**
Multiplication	$\times$ $\cdot$ $(\ )$	3×8 $3 \cdot 8$ $3(8)$	three times eight	the product of 3 and 8

Example:
Exercise:

Problem: Translate from math notation to words:

ⓐ 7×6
ⓑ $12 \cdot 14$
ⓒ $6(13)$

Solution:
Solution

- ⓐ We read this as *seven times six* and the result is *the product of seven and six*.
- ⓑ We read this as *twelve times fourteen* and the result is *the product of twelve and fourteen*.
- ⓒ We read this as *six times thirteen* and the result is *the product of six and thirteen*.

Note:
Exercise:

Problem: Translate from math notation to words:

ⓐ 8×7
ⓑ $18 \cdot 11$

Solution:

ⓐ eight times seven ; the product of eight and seven
ⓑ eighteen times eleven ; the product of eighteen and eleven

Note:
Exercise:

Problem: Translate from math notation to words:

ⓐ $(13)(7)$
ⓑ $5(16)$

Solution:

ⓐ thirteen times seven ; the product of thirteen and seven
ⓑ five times sixteen; the product of five and sixteen

Model Multiplication of Whole Numbers with Counters

There is more than one way to model multiplication. Here we will use counters to help us understand the meaning of multiplication. A counter is any object that can be used for counting. We will use round blue circles or disks.

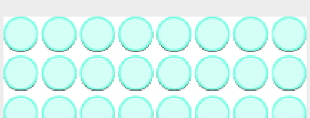
Example:
Exercise:

Problem: Model: 3×8.

Solution:
Solution

To model the product 3×8, we'll start with a row of 8 counters.

The other factor is 3, so we'll make 3 rows of 8 counters.



Count the result, or more efficiently add $8 + 8 + 8$. There are 24 counters in all.

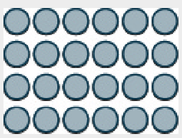
$3 \times 8 = 24$

If you look at the counters sideways, you'll see that we could have also made 8 rows of 3 counters. The product would have been the same. We'll return to this idea later.

Note:
Exercise:

Problem: Model each multiplication: 4×6.

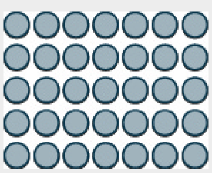
Solution:



Note:
Exercise:

Problem: Model each multiplication: 5×7.

Solution:



Multiply Whole Numbers

In order to multiply without using models, it helps to know all the one digit multiplication facts and how to multiply by 10.

The table below shows the multiplication facts. Each box shows the product of the number down the left column and the number across the top row. If you are unsure about a product, model it. It is important that you either memorize any number fact you do not already know or have a strategy for quickly getting it.

×	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

A multiplication table is a way of organizing multiplication facts so that we can easily see patterns in the facts. Since multiplication can be computed by repetitive addition, that is one way to make the table. For example, the 5s column starts, "5, 10, 15, …" These are the results when we sum 1 five, 2 fives, 3 fives,etc. We can get the next value by adding another 5, and this continues until the end of the chart. We could do this for any column. We could also do this for any row.

Note:
Multiplication Property of Zero
What happens when you multiply a number by zero? That is so easy to remember that it was not put in the chart. Thinking of multiplication as repeated addition, if the addends are zero then no matter how many times it is added, the sum is zero. Also, if the number of addends is zero then the sum is zero. This gives:
The product of any number and 0 is 0.
Equation:

$$a \cdot 0 = 0$$
$$0 \cdot a = 0$$

Example:
Exercise:

Problem: Multiply:

ⓐ $0 \cdot 11$
ⓑ $(42)0$

Solution:
Solution

ⓐ	$0 \cdot 11$
The product of any number and zero is zero.	0
ⓑ	$(42)0$
Multiplying by zero results in zero.	0

Note:
Exercise:

Problem: Find each product:

ⓐ $0 \cdot 19$
ⓑ $(39)0$

Solution:

ⓐ 0
ⓑ 0

Note:
Exercise:

Problem: Find each product:

ⓐ $0 \cdot 24$
ⓑ $(57)0$

Solution:

ⓐ 0
ⓑ 0

What happens when you multiply a number by one? Multiplying a number by one does not change the value. For this reason, 1 is called the Multiplicative Identity. An identity for an operation is a number that does not change the value of the other number in the operation. Similarly, 0 is the Additive Identity because adding 0 does not change the value.

Note:
Identity Property of Multiplication
The product of any number and 1 is the number.
Equation:

$$1 \cdot a = a$$
$$a \cdot 1 = a$$

Why is this property true? The two equations have different justifications.

- $1 \cdot a$ interpreted as repeated addition means having one addend, a, nothing is added to it, so it equals a.
- $a \cdot 1$ interpreted as repeated addition means having a addends all equal to 1. Since there are a 1s, as they are summed we get the counting numbers up to and ending with a, so that also equals a.

Example:
Exercise:

Problem: Multiply:

ⓐ $(11)1$
ⓑ $1 \cdot 42$

Solution:
Solution

ⓐ	$(11)1$
The product of any number and one is the number.	11
ⓑ	$1 \cdot 42$
Multiplying by one does not change the value.	42

Note:
Exercise:

Problem: Find each product:

ⓐ $(19)1$
ⓑ $1 \cdot 39$

Solution:

ⓐ 19
ⓑ 39

Note:
Exercise:

Problem: Find each product:

ⓐ$(24)(1)$
ⓑ $1\ \times\ 57$

Solution:

ⓐ 24
ⓑ 57

Previously, we learned that the Commutative Property of Addition states that changing the order of addition does not change the sum. We saw that $8 + 9 = 17$ gave the same sum as $9 + 8 = 17$.

Is this also true for multiplication? Let’s look at a few pairs of factors.
Equation:

$$3 \cdot 8 = 24 \qquad 8 \cdot 3 = 24$$

Equation:

$$9 \cdot 7 = 63 \qquad 7 \cdot 9 = 63$$

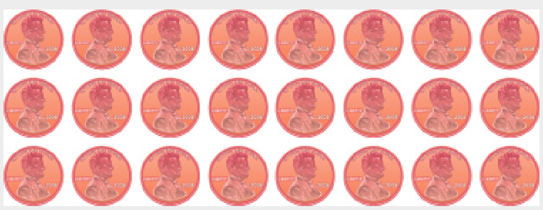
Equation:

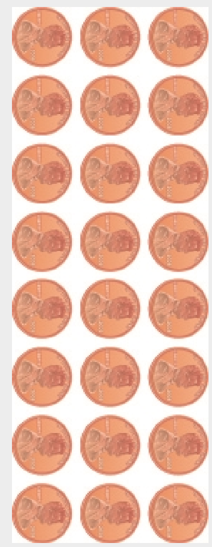
$$8 \cdot 9 = 72 \qquad 9 \cdot 8 = 72$$

Note:

Commutative Property of Multiplication
When the order of the factors is reversed, the product does not change. Recall the picture of 3 x 8 from the beginning of this section. If that is rotated 90 degrees (one quarter turn) in either direction it will look like 8 x 3. Notice that the number of counters remains the same. This is the case for any factors a x b, so a x b must equal b x a.

This property is very good news if you are learning your multiplication facts because you do not have to remember both orders of two factors. For example, 9 x 7 and 7 x 9 are both 63.





Equation:

$$a \cdot b = b \cdot a$$

Example:
Exercise:

Problem: Multiply:

ⓐ $8 \cdot 7$
ⓑ $7 \cdot 8$

Solution:
Solution

ⓐ	$8 \cdot 7$
Multiply.	56
ⓑ	$7 \cdot 8$
Multiply.	56

Changing the order of the factors does not change the product.

Note:
Exercise:

Problem: Multiply:

ⓐ $9 \cdot 6$
ⓑ $6 \cdot 9$

Solution:

54 and 54; both are the same.

Note:
Exercise:

Problem: Multiply:

ⓐ $8 \cdot 6$
ⓑ $6 \cdot 8$

Solution:

48 and 48; both are the same.

Multiplying by 10

Whenever we multiply with one factor a 10 the product has a 0 as the last digit and the preceding digits are the other factor. For example 10 x 4 = 40 and 4 x 10 = 40. Why do products like these always have a last digit of 0?

If we can find a reason for either one of these then the other must also be true because of the Commutative Property of Multiplication.
Thought of as repeated addition: 10 x 4 = 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4.

Doing the addition gives the following running total: 4, 8, 12, 16, 20, 24, 28, 32, 36, 40.
This is both the 4 row and column of the multiplication table. This doesn't really help us understand why the last digit is 0.

But looking at 4 x 10 does: 10 + 10 + 10 + 10
This has the running total 10, 20, 30, 40. Every step of the way the total ended in 0 in the ones place. When the addend is 10 there is 0 in the ones place. Since all of the addends are 10, all of those zeros will still give a 0 in the ones place.
The tens place has exactly the other factor number of tens. In this case 4. So the tens place is 4, giving a final answer of 40.

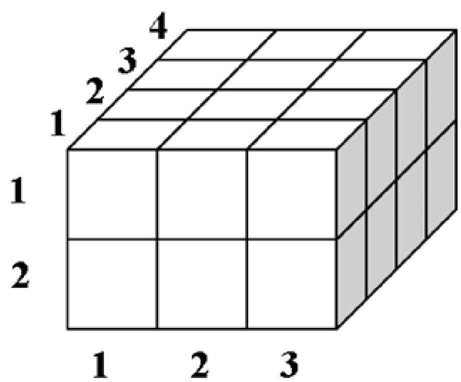
This works for larger numbers as well. For example 23 x 10 = 230. Adding twenty-three 10s will give 0 in the ones place. And twenty-three 1s in the tens place will cause the 2 to carry to the hundreds place and the 3 to remain in the tens place. In general, multiplying by 10 is easy because the digits of the product are the digits of the other factor with them all moved left to one higher place, and a 0 in the ones place.

Associative Property of Multiplication

Just like addition, multiplication is associative. That means that no matter how three factors are grouped, the result should be the same. This is claiming that:

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Here is an example. When a = 2, b = 3, and c = 4: (2 x 3) x 4 = 6 x 4 = 24 while 2 x (3 x 4) = 2 x 12 = 24. It works, at least for these numbers.



When we multiply the 2 by 3 we get 6. This is the number of cubes on just the front face of the picture. When that result multiplies 4, we are taking that entire row of 4 cubes and duplicating it 6 times. These rows of 4 are arranged in a 2 by 3 grid. This gives 24 cubes.

When we calculate 2 x (3 x 4), the first part is 3 x 4 which gives 12. For example, the 12 cubes on top. When we multiply that result by 2, that entire layer of 12 cubes is duplicated so that there are 2 levels each with 12 cubes. This also gives 24 cubes.

There was nothing special about the numbers 2, 3, and 4 other than they are small numbers so it was easy to illustrate. The idea works no matter how big the numbers. This supports the claim that the Associative Property of Multiplication is true in general. Notice that we had three factors, a, b, and c, for this property and that the illustration arranged the cubes (instead of counters) in a 3-dimensional grid. All we did was look at those cubes from different perspectives to see different ways to to calculate the total number which

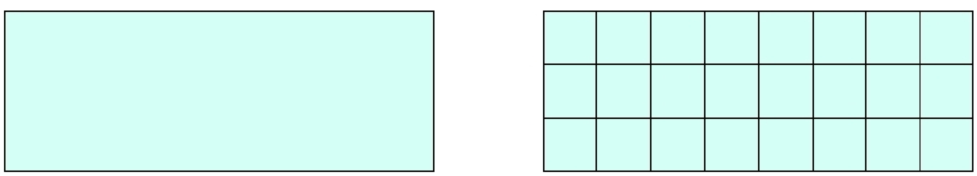
did not change. We did a similar thing in 2-dimensions for the commutative property when we looked at the counters after turning the array sideways.

Combined, the commutative and associative properties for multiplication give a result similar to the commutative and associative properties of addition. We can rearrange factors and multiply in any order we want, and we will get the same product. For example, 2 x 4 x 7 x 9 x 0 gives the same product as 0 x 2 x 4 x 7 x 9. Written the first way, we have to do 3 multiplications before finally multiplying by 0 and getting the answer 0. Written the second way we immediately see that the product is going to be 0 and thus have less work to do.

Area Model of Multiplication

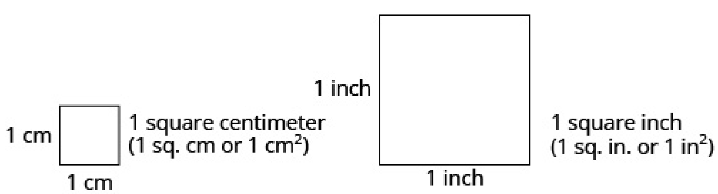
We can think of multiplication as the **area** of a rectangle rather than in terms of counters. The area is a measure of the amount of surface that is covered by the shape. It can be thought of as the number of unit squares that completely and exactly cover the figure with no overlap.

Consider a rectangle that is 8 centimeters by 3 centimeters. If we were to cover the figure with little unit squares we could think of the units as counters and the number needed would equal the length times the width.



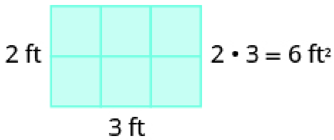
The area way of thinking about the rectangle and the counter way of thinking about the rectangle give us the same answer. Sometimes it will be easier to think in terms of area rather than having to think about counters. Other times counters will still be easier. In mathematics it is helpful to have more than one model so you can pick which ever one works best for a particular problem.

If we want to know the size of a wall that needs to be painted or a floor that needs to be carpeted, we will need to find its area. Area is measured in square units. All four sides are the same length for a square and the figure has four equal corners. Any unit of length can be used to make a square. We often use square inches, square feet, square centimeters, or square miles to measure area. A square centimeter is a square that is one centimeter (cm.) on a side. A square inch is a square that is one inch on each side, and so on.



For a rectangular figure, the area is the product of the length and the width. [link] shows a rectangular mat with a length of 2 feet and a width of 3 feet. Each square is 1 foot wide by 1 foot long, or 1 square foot. The mat can be covered completely with no overlap with 6 squares. The area of the mat is 6 square feet.

Because the squares are arranged in columns and rows of equal size, rather than count the squares or add the rows or columns, we can multiply to get the total amount. Therefore the area of a rectangle is its length x width: A = L x W.



The area of a rectangle is the product of its length and its width, or 6 square feet.

Distributive Property of Multiplication over Addition

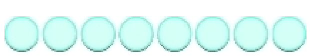
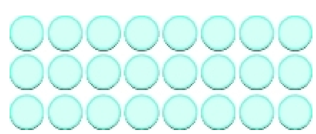
The Distributive Property of Multiplication over Addition is our first property that involves more than one operation. Normally it is just called the Distributive Property. The reason for the long name is that in more advanced math there are other distributive properties.

$a \cdot (b + c) = a \cdot b + a \cdot c$

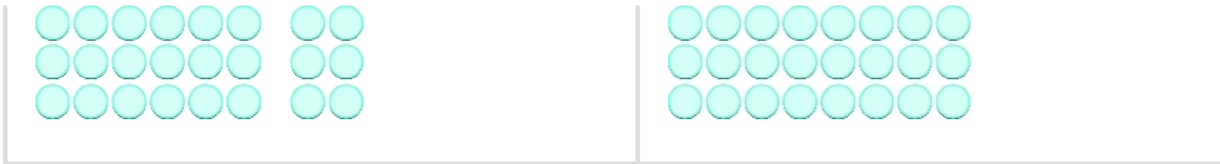
To help get a feeling for why this property is true, we'll use some numbers to see how it works.
For example, let a=3, b=6, and c=2. Then the equation becomes: 3 x (6 + 2) = (3 x 6) + (3 x 2)

The left hand side of the equation computes: 3 x (6 + 2) = 3 x 8 = 24.
The right hand side of the equation computes: (3 x 6) + (3 x 2) = 18 + 6 = 24. At least in this case, the property is true.

We can use counters to illustrate each side of the equation. The left hand side of the equation has one row of six counters with two counters added to it making one row of 8 counters. This is then multiplied to make 3 rows of 8 counters.

The right hand side of the equation starts with 3 rows of 6 counters followed by 3 rows of 2 counters. They are then added resulting in 3 rows of 8 counters.



In both cases that we end up with 24 counters arranged as 3 rows of 8.

This works for any number of rows no matter how long the rows are as long as they are all same length.

Multiplying Multi-digit Numbers

With the exception of multiplying by 10, all of the multiplication we've done so far has been a single digit times a single digit. We have built a multiplication table that organizes and holds those facts and either we can recall them automatically or have a strategy for quickly recalling them.

What about multi-digit multiplication? Should we enlarge the multiplication table for numbers greater than 10? We could, but this isn't a good strategy because the table would quickly either become very, very large or else not be big enough.

Because our numbers are represented using place value this isn't necessary. We can master a procedure for multiplying large numbers that doesn't require learning any more facts than the basic ones we already know.

There is more than one correct way to do multi-digit multiplication. We will look at two related ways. The first way you might have learned in school. We'll call it the Compact Method or Standard Method. The second way uses the Area Model of Multiplication and so we'll call it the Area Method. It has the benefit of being easier to find your mistakes if you happen to make one. Let's look at the more common way first.

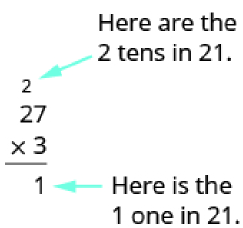
To multiply numbers with more than one digit, it is usually easier to write the numbers vertically in columns just as we did for addition and subtraction. It is also usually easier to write the smaller number with fewer digits on the bottom.
Equation:

$$\begin{array}{r} 27 \\ \underline{\times 3} \end{array}$$

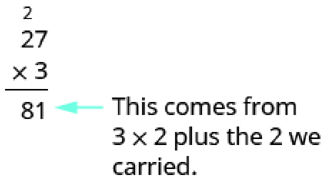
We start by multiplying 3 by 7.
Equation:

$$3 \times 7 = 21$$

We write the 1 in the ones place of the product. We carry the 2 tens by writing 2 above the tens place.



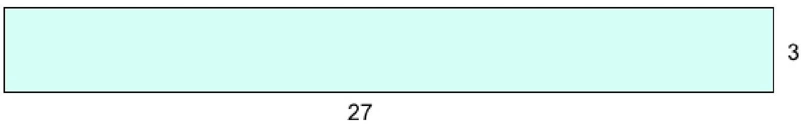
Then we multiply the 3 by the 2, and add the 2 above the tens place to the product. So $3 \times 2 = 6$, and $6 + 2 = 8$. Write the 8 in the tens place of the product.



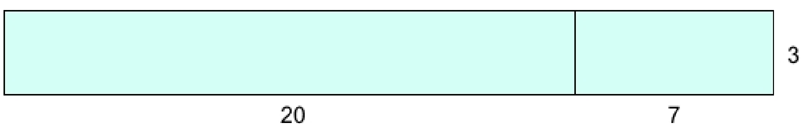
The product is 81.

The Area Model of Multiplication

Let's look at the same problem using the Area Model of Multiplication. Start by drawing a 27 by 3 rectangle. Occasionally, when we draw diagrams in math it helps to have the sizes fairly accurate. This is NOT one of those times, so don't worry about having the lengths perfect in relationship to each other.



The key to this method is to split any side that is has more than one digit into parts based on the Expanded Form for the number. In our example, we split 27 into 20 and 7. Because it is more familiar for us to have the 20 on the left and the 7 on the right, it is easiest if we do it that way. The other dimension, 3, is unchanged. Now we have two smaller rectangles rather than just one big rectangle.



We can find 27 x 3 if we can find the area of the 27 by 3 rectangle.
We can find the area of the large rectangle by finding the area of the two smaller rectangles and then adding them together.
This uses Expanded Form and then the Distributive Property.

3 x 27 = 3 x (20 + 7) = (3 x 20) + (3 x 7).

Calculating 3 x 7 is easy, it is just a basic multiplication fact. Calculating 3 x 20 requires a little more work.
You may already know that 3 x 20 = 60.
Let's see how we know that using the properties of multiplication and the multiplication table.

Any number whose last digit is a "0" is the product of the number that has all the same digits in the same order but without the final "0" times 10.
In this case, 20 = 2 x 10. "2" has the same digits as "20" except for the final "0".

Substituting 2 x 10 for 20 gives: 3 x (2 x 10).
Using the Associative Property of Multiplication yields: (3 x 2) x 10.
Using the multiplication table gives: 6 x 10.
Finally using the Multiplication Property of 10 gives 60.

To finish the problem, we need to calculate 60 + 21 = 81.

No need to draw a diagram to multiply larger numbers. This illustration that shows why the method works. It is recommended that you write:

$$\begin{array}{r} 27 \\ \times\ 3 \\ \hline 21 \\ +60 \\ \hline 81 \end{array}$$

This is only a tiny bit more writing than the compact method. It has an advantage of separating the recall of multiplication facts from the addition.
Each multiplication is written down and then addition is performed. This makes it easier to check your work.
It also shows the connection between those products and the smaller rectangles that make up the large rectangle.
In our example, we see the 3 x 7 = 21 rectangle and the 3 x 20 = 60 rectangle combine to make the 3 x 27 = 81 rectangle.

Example:
Exercise:

Problem: Multiply: $15 \cdot 4$.

Solution:
Compact Method

Write the numbers so the digits 5 and 4 line up vertically.	$\begin{array}{r} 15 \\ \times\ 4 \\ \hline \end{array}$
Multiply 4 by the digit in the ones place of 15. $4 \cdot 5 = 20$.	
Write 0 in the ones place of the product and carry the 2 tens.	$\begin{array}{r} \overset{2}{1}5 \\ \times\ 4 \\ \hline 0 \end{array}$
Multiply 4 by the digit in the tens place of 15. $4 \cdot 1 = 4$. Add the 2 tens we carried. $4 + 2 = 6$.	
Write the 6 in the tens place of the product.	$\begin{array}{r} \overset{2}{1}5 \\ \times\ 4 \\ \hline 60 \end{array}$

Try to do this problem again using the area method. What large rectangle would you draw? What two rectangles would you break it up into?
Using the area model, what numbers should you write and how should they be organized?

Note:
For this problem, think about the dimensions of the two small rectangles but don't draw them. Instead write their areas and add them for the final answer.
Exercise:

Problem: Multiply: $64 \cdot 8$.
One rectangle is 4 x 8 = 32 and the other is 60 x 8 = 480.

Solution:

512

Note:
Exercise:

Problem: Multiply: $57 \cdot 6$.

Solution:

342

Example:
Exercise:

Problem: Multiply: $286 \cdot 5$.

Solution:
Compact Method

Write the numbers so the digits 5 and 6 line up vertically.	$\begin{array}{r} 286 \\ \underline{\times\ 5} \end{array}$
Multiply 5 by the digit in the ones place of 286. $5 \cdot 6 = 30$.	
Write the 0 in the ones place of the product and carry the 3 to the tens place.Multiply 5 by the digit in the tens place of 286. $5 \cdot 8 = 40$.	

	$\begin{array}{r} \overset{3}{2}86 \\ \times\ 5 \\ \hline 0 \end{array}$
Add the 3 tens we carried to get $40 + 3 = 43$. Write the 3 in the tens place of the product and carry the 4 to the hundreds place.	$\begin{array}{r} \overset{4\ 3}{286} \\ \times\ 5 \\ \hline 30 \end{array}$
Multiply 5 by the digit in the hundreds place of 286. $5 \cdot 2 = 10$. Add the 4 hundreds we carried to get $10 + 4 = 14$. Write the 4 in the hundreds place of the product and the 1 to the thousands place.	$\begin{array}{r} \overset{4\ 3}{286} \\ \times\ 5 \\ \hline 1{,}430 \end{array}$

Do this problem using the area model. Since there are 3 digits in the larger number, the large rectangle will be broken up into three smaller rectangles. Draw them and then complete the multiplication. Hint #1: Use the Expanded Form for 286 = 200 + 80 + 6. Hint #2: 200 = 20 x 10 = 2 x 10 x 10.

Note:
Exercise:

Problem: Multiply: $347 \cdot 5$.

Solution:

$1{,}735$

Note:
Exercise:

Problem: Multiply: $462 \cdot 7$.

Solution:

$3{,}234$

When we multiply by a number with two or more digits, we multiply by each of the digits separately, working from right to left. Each separate product of the digits is called a partial product. When we write partial products, we must make sure to line up the place values.

Note:
Multiply two whole numbers to find the product using the Compact Method

Write the numbers so each place value lines up vertically.

Multiply the digits in each place value.

- Work from right to left, starting with the ones place in the bottom number.
  - Multiply the bottom digit by the ones digit in the top number, then by the tens digit, and so on.
  - If a product in a place value is more than 9, carry to the next place value.
  - Write the partial products, lining up the digits in the place values with the numbers above.
- Repeat for the tens place in the bottom number, the hundreds place, and so on.
- Insert a zero as a placeholder with each additional partial product.

Add the partial products.

Example:
Exercise:

Problem: Multiply: $62(87)$.

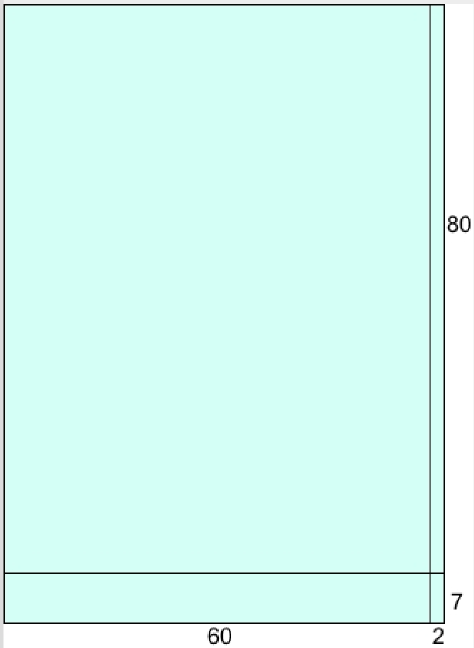
Solution:
Solution

Write the numbers so each place lines up vertically.	
Start by multiplying 7 by 62. Multiply 7 by the digit in the ones place of 62. $7 \cdot 2 = 14$. Write the 4 in the ones place of the product and carry the 1 to the tens place.	
Multiply 7 by the digit in the tens place of 62. $7 \cdot 6 = 42$. Add the 1 ten we carried. $42 + 1 = 43$. Write the 3 in the tens place of the product and the 4 in the hundreds place.	
The first partial product is 434.	
Now, write a 0 under the 4 in the ones place of the next partial product as a placeholder since we now multiply the digit in the tens place of 87 by 62. Multiply 8 by the digit in the ones place of 62. $8 \cdot 2 = 16$. Write the 6 in the next place of the product, which is the tens place. Carry the 1 to the tens place.	
Multiply 8 by 6, the digit in the tens place of 62, then add the 1 ten we carried to get 49. Write the 9 in the hundreds place of the product and the 4 in the thousands place.	

The second partial product is 4960. Add the partial products.

The product is 5,394.

Let's look at the same problem using the Area Model of Multiplication. Since both factors have multiple digits, both sides of the 62 by 87 rectangle needed to be broken into pieces using Expanded Form. One side will be 60 + 2 and the other side will be 80 + 7.



Find the area of all four rectangles. This amounts to doing 4 multiplications:
2 x 7,
60 x 7,
2 x 80,
60 x 80.

$$\begin{array}{r} 14 \\ 420 \\ 160 \\ \underline{+\ 4800} \\ 5394 \end{array}$$

Note:
Multiply Two Whole Numbers to Find the Product Using the Area Model

- Write the numbers so each place value lines up vertically.
- Multiply the digits in each place value.
    - Work from right to left, starting with the ones place in the bottom number.

        - Write the partial product, lining up the digits in the place values with the numbers above.
        - Write one zero on the right for each place past the ones place for both the top and bottom digits.
        - Multiply the bottom digit by the ones digit in the top number, then by the tens digit, and so on.
        - Write that product to the left of any zeros.

    - Repeat for the tens place in the bottom number, the hundreds place, and so on.

- Add the partial products.

$$\begin{array}{r} 62 \\ \times\ 87 \\ \hline 14 \\ 420 \\ 160 \\ +\ 4800 \\ \hline 5394 \end{array}$$

Note:
Exercise:

Problem: Multiply: $43(78)$.

Solution:

3,354

Note:
Exercise:

Problem: Multiply: $64(59)$.

Solution:

3,776

Example:
Exercise:

Problem: Multiply:

ⓐ $47 \cdot 10$
ⓑ $47 \cdot 100$.

Solution:
Solution

When we multiply 47 times 10, the product is 470. Notice that 10 has one zero, and we put the zero directly to the right of the 47 to get the product. When we multiply 47 times 100, the product is 4,700. Notice that 100 has two zeros and we put two zeros after 47 to get the product.

Do you see the pattern? If we multiplied 47 times 10,000, which has four zeros, we would put four zeros after 47 to get the product 470,000.

Why does this work?
47 x 10,000 = 47 x (10 x 1,000)
47 x (10 x 1,000) = (47 x 10) x 1,000
(47 x 10) x 1,000 = 470 x 1,000
470 x 1,000 = 470 x (10 x 100)
470 x (10 x 100) = (470 x 10) x 100
(470 x 10) x 100 = 4,700 x 100
4,700 x 100 = 4,700 x (10 x 10)
4,700 x (10 x 10) = (4,700 x 10) x 10
(4,700 x 10) x 10 = 47,000 x 10
47,000 x 10 = 470,000

This repeatedly uses the Associative Property of Multiplication and our rule for Multiplying by 10.

You do not have to think of this process every time to get the correct answer.
Use the shortcut of counting the total number of zeros to the right of the first non-zero digits of both factors. For example, 210 x 300 = 63,000 because 21 x 3 = 63 and there are a total of 3 zeros prior to the non-zero digits.

Note:
Exercise:

Problem: Multiply:

ⓐ $54 \cdot 10$
ⓑ $54 \cdot 100$.

Solution:

ⓐ 540
ⓑ 5,400

Note:
Exercise:

Problem: Multiply:

ⓐ $75 \cdot 10$
ⓑ $75 \cdot 100.$

Solution:

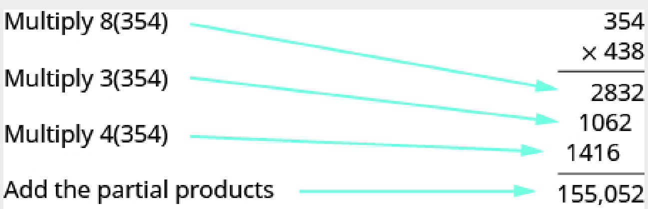
ⓐ 750
ⓑ 7,500

Example:
Exercise:

Problem: Multiply: $(354)(438).$

Solution:
Solution

There are three digits in the factors so there will be 3 partial products. We do not have to write the 0 as a placeholder as long as we write each partial product in the correct place.



Doing this problem with the Area Model of Multiplication involves finding 9 areas and then adding them together. Draw a rectangle and break it up into the 9 pieces. Find each product and then find the sum. Check your answer against the solution above.

Note:
Exercise:

Problem: Multiply: $(265)(483).$

Solution:

127,995

Note:

Exercise:

Problem: Multiply: $(823)(794)$.

Solution:

653,462

Example:
Exercise:

Problem: Multiply: $(896)201$.

Solution:
Solution

There should be 3 partial products. The second partial product will be the result of multiplying 896 by 0.

Notice that the second partial product of all zeros doesn’t really affect the result. We can place a zero as a placeholder in the tens place and then proceed directly to multiplying by the 2 in the hundreds place, as shown.

Multiply by 10, but insert only one zero as a placeholder in the tens place. Multiply by 200, putting the 2 from the $12.$ $2\cdot 6=12$ in the hundreds place.
Equation:

$$\begin{array}{r} 896\\ \underline{\times 201}\\ 896\\ 17920\\ \hline 180{,}096 \end{array}$$

Do this problem with the Area Model of Multiplication. The second factor, 201, has 0 in the tens place and so that side's Expanded Form will only have 2 pieces. Therefore there will be a total of 6 small rectangles. Make sure to label the side with 200 and not 20 or else your area for that piece will be off by a factor of 10.
Draw the 6 rectangles putting 896 horizontal (800, 90, 6) and 201 vertical (200, 1). Obviously not to scale. Show the six products, one line each, and then the sum.

Note:
Exercise:

Problem: Multiply: $(718)509$.

Solution:

365,462

Note:
Exercise:

Problem: Multiply: $(627)804$.

Solution:

504,108

When there are three or more factors, we multiply the first two and then multiply their product by the next factor. For example:

to multiply	$8 \cdot 3 \cdot 2$
first multiply $8 \cdot 3$	$24 \cdot 2$
then multiply $24 \cdot 2$.	48

Since multiplication is commutative and associative, we could multiply the factors in any order and get the same answer. It might be easier to multiply the 3 x 2 = 6 first and then 8 x 6 = 48 since all of these multiplications are single digit facts.

Translate Word Phrases to Math Notation

Earlier in this section, we translated math notation into words. Now we'll reverse the process and translate word phrases into math notation. Some of the words that indicate multiplication are given in [link].

Operation	Word Phrase	Example	Expression
Multiplication	times product twice	3 times 8 the product of 3 and 8 twice 4	$3\ \times\ 8, 3 \cdot 8, (3)(8),$ $(3)8,$ or $3(8)$ $2 \cdot 4$

Example:
Exercise:

Problem: Translate and simplify: the product of 12 and 27.

Solution:
Solution

The word *product* tells us to multiply. The words *of* 12 *and* 27 tell us the two factors.

	the product of 12 and 27
Translate.	$12 \cdot 27$
Multiply.	324

Note:
Exercise:

Problem: Translate and simplify the product of 13 and 28.

Solution:

$13 \cdot 28$; 364

Note:
Exercise:

Problem: Translate and simplify the product of 47 and 14.

Solution:

47 · 14; 658

Example:
Exercise:

Problem: Translate and simplify: twice two hundred eleven.

Solution:
Solution

The word *twice* tells us to multiply by 2.

	twice two hundred eleven
Translate.	2(211)
Multiply.	422

Note:
Exercise:

Problem: Translate and simplify: twice one hundred sixty-seven.

Solution:

2(167); 334

Note:
Exercise:

Problem: Translate and simplify: twice two hundred fifty-eight.

Solution:

2(258); 516

Multiply Whole Numbers in Applications

We will use the same strategy we used previously to solve applications of multiplication.

- Determine what we are asked to find.
- Write a phrase that gives the information to find it.
- Translate the phrase into math notation.
- Simplify to get the answer.
- Write a sentence to answer the question, using the appropriate units.

Example:
Exercise:

Problem:

Humberto bought 4 sheets of stamps. Each sheet had 20 stamps. How many stamps did Humberto buy?

Solution:
Solution

We are asked to find the total number of stamps.

Write a phrase for the total.	the product of 4 and 20
Translate to math notation.	$4 \cdot 20$
Multiply.	$\begin{array}{r} 20 \\ \times 4 \\ \hline 80 \end{array}$
Write a sentence to answer the question.	Humberto bought 80 stamps.

Note:
Exercise:

Problem:

Valia donated water for the snack bar at her son's baseball game. She brought 6 cases of water bottles. Each case had 24 water bottles. How many water bottles did Valia donate?

Solution:

Valia donated 144 water bottles.

Note:
Exercise:

Problem:

Vanessa brought 8 packs of hot dogs to a family reunion. Each pack has 10 hot dogs. How many hot dogs did Vanessa bring?

Solution:

Vanessa bought 80 hot dogs.

Example:
Exercise:

Problem:

When Rena cooks rice, she uses twice as much water as rice. How much water does she need to cook 4 cups of rice?

Solution:
Solution

We are asked to find how much water Rena needs.

Write as a phrase.	twice as much as 4 cups
Translate to math notation.	$2 \cdot 4$
Multiply to simplify.	8
Write a sentence to answer the question.	Rena needs 8 cups of water for cups of rice.

Note:
Exercise:

Problem:

Erin is planning her flower garden. She wants to plant twice as many dahlias as sunflowers. If she plants 14 sunflowers, how many dahlias does she need?

Solution:

Erin needs 28 dahlias.

Note:
Exercise:

Problem:

A college choir has twice as many women as men. There are 18 men in the choir. How many women are in the choir?

Solution:

There are 36 women in the choir.

Example:
Exercise:

Problem:

Van is planning to build a patio. He will have 8 rows of tiles, with 14 tiles in each row. How many tiles does he need for the patio?

Solution:
Solution

We are asked to find the total number of tiles.

Write a phrase.	the product of 8 and 14
Translate to math notation.	$8 \cdot 14$
Multiply to simplify.	$\begin{array}{r} \overset{3}{1}4 \\ \underline{\times 8} \\ 112 \end{array}$

Write a sentence to answer the question.	Van needs 112 tiles for his patio.

Note:
Exercise:

Problem:

Jane is tiling her living room floor. She will need 16 rows of tile, with 20 tiles in each row. How many tiles does she need for the living room floor?

Solution:

Jane needs 320 tiles.

Note:
Exercise:

Problem:

Yousef is putting shingles on his garage roof. He will need 24 rows of shingles, with 45 shingles in each row. How many shingles does he need for the garage roof?

Solution:

Yousef needs 1,080 tiles.

Example:
Exercise:

Problem:

Jen’s kitchen ceiling is a rectangle that measures 9 feet long by 12 feet wide. What is the area of Jen’s kitchen ceiling?

Solution:
Solution

We are asked to find the area of the kitchen ceiling.

Write a phrase for the area.	the product of 9 and 12
Translate to math notation.	$9 \cdot 12$

Multiply.	$\begin{array}{r} \overset{1}{1}2 \\ \underline{\times 9} \\ 108 \end{array}$
Answer with a sentence.	The area of Jen's kitchen ceiling is 108 square feet.

Note:
Exercise:

Problem:

Zoila bought a rectangular rug. The rug is 8 feet long by 5 feet wide. What is the area of the rug?

Solution:

The area of the rug is 40 square feet.

Note:
Exercise:

Problem:

Rene’s driveway is a rectangle 45 feet long by 20 feet wide. What is the area of the driveway?

Solution:

The area of the driveway is 900 square feet

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Area
- Multiplying Whole Numbers
- Multiplication with Partial Products
- Example of Multiplying by Whole Numbers

Key Concepts

Operation	Notation	Expression	Read as	Result
Multiplication	$\times$ $\cdot$ ( )	3×8 $3 \cdot 8$ $3(8)$	three times eight	the product of 3 and 8

- **Multiplication Property of Zero**
  - The product of any number and 0 is 0.
    $a \cdot 0 = 0$
    $0 \cdot a = 0$
- **Identity Property of Multiplication**
  - The product of any number and 1 is the number.
    $1 \cdot a = a$
    $a \cdot 1 = a$
- **Commutative Property of Multiplication**
  - Changing the order of the factors does not change their product.
    $a \cdot b = b \cdot a$
- **Multiplying by 10**
  - Multiplying by a factor of 10 results in the same digits immediately followed by a digit of zero.
- **Associative Property of Multiplication**
  - Changing the grouping of the factors does not change their product.
    $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- **Distributive Property of Multiplication over Addition**
  - A factor multiplying a sum can be changed to the sum of the factor multiplying each addend.
    $a \cdot (b + c) = a \cdot b + a \cdot c$
- **Multiply two whole numbers to find the product.**

  Write the numbers so each place value lines up vertically.
  Multiply the digits in each place value.
  Work from right to left, starting with the ones place in the bottom number.
  Multiply the bottom number by the ones digit in the top number, then by the tens digit, and so on.
  If a product in a place value is more than 9, carry to the next place value.
  Write the partial products, lining up the digits in the place values with the numbers above. Repeat for the tens place in the bottom number, the hundreds place, and so on.
  Insert a zero as a placeholder with each additional partial product.
  Add the partial products.

Practice Makes Perfect

Use Multiplication Notation

In the following exercises, translate from math notation to words.

Exercise:

Problem: 4×7

Solution:

four times seven; the product of four and seven

Exercise:

Problem: 8×6

Exercise:

Problem: $5 \cdot 12$

Solution:

five times twelve; the product of five and twelve

Exercise:

Problem: $3 \cdot 9$

Exercise:

Problem: $(10)(25)$

Solution:

ten times twenty-five; the product of ten and twenty-five

Exercise:

Problem: $(20)(15)$

Exercise:

Problem: $42(33)$

Solution:

forty-two times thirty-three; the product of forty-two and thirty-three

Exercise:

Problem: $39(64)$

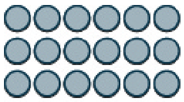
Model Multiplication of Whole Numbers

In the following exercises, model the multiplication.

Exercise:

Problem: 3×6

Solution:



$3 \times 6 = 18$

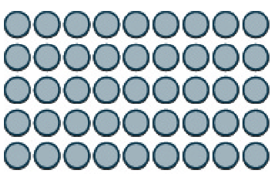
Exercise:

Problem: 4×5

Exercise:

Problem: 5×9

Solution:



$5 \times 9 = 45$

Exercise:

Problem: 3×9

Multiply Whole Numbers

In the following exercises, fill in the missing values in each chart.

Exercise:

Problem:

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0		0	0	0	0		0
1	0	1	2	3			6	7	8	
2		2	4	6	8		12			18
3	0		6		12	15		21		27
4	0	4			16	20		28	32	
5	0	5	10	15			30		40	
6	0	6	12		24			42		54
7			14	21		35			56	63
8	0	8		24			48		64	
9	0	9	18		36	45			72	

Solution:

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Exercise:

Problem:

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0		0		0	0
1	0	1	2		4	5		7		9
2	0		4		8	10		14	16	
3		3		9			18		24	
4	0	4	8	12			24	28		36
5	0	5		15	20		30	35	40	
6			12	18			36	42		54
7	0	7		21		35			56	63
8	0	8	16		32		48		64	72
9			18	27	36			63		

Exercise:

Problem:

×	3	4	5	6	7	8	9
4							
5							
6							
7							
8							
9							

Solution:

×	3	4	5	6	7	8	9
4	12	16	20	24	28	32	36
5	15	20	25	30	35	40	45
6	18	24	30	36	42	48	54
7	21	28	35	42	49	56	63
8	24	32	40	48	56	64	72
9	27	36	45	54	63	72	81

Exercise:

Problem:

×	4	5	6	7	8	9
3						
4						
5						
6						
7						
8						
9						

Exercise:

Problem:

×	3	4	5	6	7	8	9
6							
7							
8							
9							

Solution:

×	3	4	5	6	7	8	9
6	18	24	30	36	42	48	54
7	21	28	35	42	49	56	63
8	24	32	40	48	56	64	72
9	27	36	45	54	63	72	81

Exercise:

Problem:

×	6	7	8	9
3				
4				
5				
6				
7				
8				
9				

Exercise:

Problem:

×	5	6	7	8	9
5					
6					
7					
8					
9					

Solution:

×	5	6	7	8	9
5	25	30	35	40	45
6	30	36	42	48	54
7	35	42	49	56	63
8	40	48	56	64	72
9	45	54	63	72	81

Exercise:

Problem:

×	6	7	8	9
6				
7				
8				
9				

In the following exercises, multiply.

Exercise:

Problem: $0 \cdot 15$

Solution:

0

Exercise:

Problem: $0 \cdot 41$

Exercise:

Problem: $(99)0$

Solution:

0

Exercise:

Problem: $(77)0$

Exercise:

Problem: $1 \cdot 43$

Solution:

43

Exercise:

Problem: $1 \cdot 34$

Exercise:

Problem: $(28)1$

Solution:

28

Exercise:

Problem: $(65)1$

Exercise:

Problem: $1(240{,}055)$

Solution:

240,055

Exercise:

Problem: $1(189{,}206)$

Exercise:

Problem:

ⓐ $7 \cdot 6$
ⓑ $6 \cdot 7$

Solution:

ⓐ 42
ⓑ 42

Exercise:

Problem:

ⓐ 8×9
ⓑ 9×8

Exercise:

Problem: $(79)(5)$

Solution:

395

Exercise:

Problem: $(58)(4)$

Exercise:

Problem: $275 \cdot 6$

Solution:

1,650

Exercise:

Problem: $638 \cdot 5$

Exercise:

Problem: $3{,}421 \ \times \ 7$

Solution:

23,947

Exercise:

Problem: $9{,}143 \ \times \ 3$

Exercise:

Problem: $52(38)$

Solution:

1,976

Exercise:

Problem: $37(45)$

Exercise:

Problem: $96 \cdot 73$

Solution:

7,008

Exercise:

Problem: $89 \cdot 56$

Exercise:

Problem: 27×85

Solution:

2,295

Exercise:

Problem: 53×98

Exercise:

Problem: $23 \cdot 10$

Solution:

230

Exercise:

Problem: $19 \cdot 10$

Exercise:

Problem: $(100)(36)$

Solution:

3,600

Exercise:

Problem: $(100)(25)$

Exercise:

Problem: $1{,}000(88)$

Solution:

88,000

Exercise:

Problem: $1{,}000(46)$

Exercise:

Problem: $50 \times 1{,}000{,}000$

Solution:

50,000,000

Exercise:

Problem: $30 \times 1{,}000{,}000$

Exercise:

Problem: 247×139

Solution:

34,333

Exercise:

Problem: 156×328

Exercise:

Problem: $586(721)$

Solution:

422,506

Exercise:

Problem: $472(855)$

Exercise:

Problem: $915 \cdot 879$

Solution:

804,285

Exercise:

Problem: $968 \cdot 926$

Exercise:

Problem: $(104)(256)$

Solution:

26,624

Exercise:

Problem: $(103)(497)$

Exercise:

Problem: $348(705)$

Solution:

245,340

Exercise:

Problem: $485(602)$

Exercise:

Problem: $2{,}719\ \times\ 543$

Solution:

1,476,417

Exercise:

Problem: $3{,}581\ \times\ 724$

Translate Word Phrases to Math Notation

In the following exercises, translate and simplify.

Exercise:

Problem: the product of 18 and 33

Solution:

$18 \cdot 33$; 594

Exercise:

Problem: the product of 15 and 22

Exercise:

Problem: fifty-one times sixty-seven

Solution:

51(67); 3,417

Exercise:

Problem: forty-eight times seventy-one

Exercise:

Problem: twice 249

Solution:

2(249); 498

Exercise:

Problem: twice 589

Exercise:

Problem: ten times three hundred seventy-five

Solution:

10(375); 3,750

Exercise:

Problem: ten times two hundred fifty-five

Mixed Practice

In the following exercises, simplify.

Exercise:

Problem: 38×37

Solution:

1,406

Exercise:

Problem: 86×29

Exercise:

Problem: $415 - 267$

Solution:

148

Exercise:

Problem: $341 - 285$

Exercise:

Problem: $6{,}251 + 4{,}749$

Solution:

11,000

Exercise:

Problem: $3{,}816 + 8{,}184$

Exercise:

Problem: $(56)(204)$

Solution:

11,424

Exercise:

Problem: $(77)(801)$

Exercise:

Problem: $947 \cdot 0$

Solution:

0

Exercise:

Problem: $947 + 0$

Exercise:

Problem: $15{,}382 + 1$

Solution:

15,383

Exercise:

Problem: $15{,}382 \cdot 1$

In the following exercises, translate and simplify.

Exercise:

Problem: the difference of 50 and 18

Solution:

$50 - 18$; 32

Exercise:

Problem: the difference of 90 and 66

Exercise:

Problem: twice 35

Solution:

2(35); 70

Exercise:

Problem: twice 140

Exercise:

Problem: 20 more than 980

Solution:

20 + 980; 1,000

Exercise:

Problem: 65 more than 325

Exercise:

Problem: the product of 12 and 875

Solution:

12(875); 10,500

Exercise:

Problem: the product of 15 and 905

Exercise:

Problem: subtract 74 from 89

Solution:

89 − 74; 15

Exercise:

Problem: subtract 45 from 99

Exercise:

Problem: the sum of 3,075 and 950

Solution:

3,075 + 950; 4,025

Exercise:

Problem: the sum of 6,308 and 724

Exercise:

Problem: 366 less than 814

Solution:

814 − 366; 448

Exercise:

Problem: 388 less than 925

Multiply Whole Numbers in Applications

In the following exercises, solve.

Exercise:

Problem:

Party supplies Tim brought 9 six-packs of soda to a club party. How many cans of soda did Tim bring?

Solution:

Tim brought 54 cans of soda to the party.

Exercise:

Problem:

Sewing Kanisha is making a quilt. She bought 6 cards of buttons. Each card had four buttons on it. How many buttons did Kanisha buy?

Exercise:

Problem:

Field trip Seven school busses let off their students in front of a museum in Washington, DC. Each school bus had 44 students. How many students were there?

Solution:

There were 308 students.

Exercise:

Problem:

Gardening Kathryn bought 8 flats of impatiens for her flower bed. Each flat has 24 flowers. How many flowers did Kathryn buy?

Exercise:

Problem:

Charity Rey donated 15 twelve-packs of t-shirts to a homeless shelter. How many t-shirts did he donate?

Solution:

Rey donated 180 t-shirts.

Exercise:

Problem:

School There are 28 classrooms at Anna C. Scott elementary school. Each classroom has 26 student desks. What is the total number of student desks?

Exercise:

Problem:

Recipe Stephanie is making punch for a party. The recipe calls for twice as much fruit juice as club soda. If she uses 10 cups of club soda, how much fruit juice should she use?

Solution:

Stephanie should use 20 cups of fruit juice.

Exercise:

Problem:

Gardening Hiroko is putting in a vegetable garden. He wants to have twice as many lettuce plants as tomato plants. If he buys 12 tomato plants, how many lettuce plants should he get?

Exercise:

Problem:

Government The United States Senate has twice as many senators as there are states in the United States. There are 50 states. How many senators are there in the United States Senate?

Solution:

There are 100 senators in the U.S. senate.

Exercise:

Problem:

Recipe Andrea is making potato salad for a buffet luncheon. The recipe says the number of servings of potato salad will be twice the number of pounds of potatoes. If she buys 30 pounds of potatoes, how many servings of potato salad will there be?

Exercise:

Problem:

Painting Jane is painting one wall of her living room. The wall is rectangular, 13 feet wide by 9 feet high. What is the area of the wall?

Solution:

The area of the wall is 117 square feet.

Exercise:

Problem:

Home décor Shawnte bought a rug for the hall of her apartment. The rug is 3 feet wide by 18 feet long. What is the area of the rug?

Exercise:

Problem:

Room size The meeting room in a senior center is rectangular, with length 42 feet and width 34 feet. What is the area of the meeting room?

Solution:

The area of the room is 1,428 square feet.

Exercise:

Problem:

Gardening June has a vegetable garden in her yard. The garden is rectangular, with length 23 feet and width 28 feet. What is the area of the garden?

Exercise:

Problem:

NCAA basketball According to NCAA regulations, the dimensions of a rectangular basketball court must be 94 feet by 50 feet. What is the area of the basketball court?

Solution:

The area of the court is 4,700 square feet.

Exercise:

Problem:

NCAA football According to NCAA regulations, the dimensions of a rectangular football field must be 360 feet by 160 feet. What is the area of the football field?

Everyday Math

Exercise:

Problem:

Stock market Javier owns 300 shares of stock in one company. On Tuesday, the stock price rose $12 per share. How much money did Javier's portfolio gain?

Solution:

Javier's portfolio gained $3,600.

Exercise:

Problem:

Salary Carlton got a $200 raise in each paycheck. He gets paid 24 times a year. How much higher is his new annual salary?

Writing Exercises

Exercise:

Problem:

How confident do you feel about your knowledge of the multiplication facts? If you are not fully confident, what will you do to improve your skills?

Solution:

Answers will vary.

Exercise:

Problem: How have you used models to help you learn the multiplication facts?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use multiplication notation.			
model multiplication of whole numbers.			
multiply whole numbers.			
translate word phrases to math notation.			
multiply whole numbers in applications.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

product
: The product is the result of multiplying two or more numbers.

Divide Whole Numbers Intermediate Level
By the end of this section, you will be able to:

- Use division notation
- Model division of whole numbers
- Divide whole numbers
- Translate word phrases to math notation
- Divide whole numbers in applications

Note:
Before you get started, take this readiness quiz.

1. Multiply: $27 \cdot 3$.
   If you missed this problem, review [link].
2. Subtract: $43 - 26$.
   If you missed this problem, review [link]
3. Multiply: $62(87)$.
   If you missed this problem, review [link].

Use Division Notation

So far we have explored addition, subtraction, and multiplication. Now let’s consider division. Suppose you have the 12 cookies in [link] and want to package them in bags with 4 cookies in each bag. How many bags would we need?

You might put 4 cookies in first bag, 4 in the second bag, and so on until you run out of cookies. Doing it this way, you would fill 3 bags.

In other words, starting with the 12 cookies, you would take away, or subtract, 4 cookies at a time. Division is a way to represent repeated subtraction just as multiplication represents repeated addition.

Instead of subtracting 4 repeatedly, we can write
Equation:

$$12 \div 4$$

We read this as *twelve divided by four* and the result is the **quotient** of 12 and 4. The quotient is 3 because we can subtract 4 from 12 exactly 3 times. We call the number being divided the **dividend** and the number dividing it the **divisor**. In this case, the dividend is 12 and the divisor is 4.

The word "quotient" comes from Latin and means "how many times".

In the past you may have used the notation $4\overline{)12}$, but this division also can be written as $12 \div 4,\ 12/4,\ \frac{12}{4}$. In each case the 12 is the dividend and the 4 is the divisor.

Note:
Operation Symbols for Division
To represent and describe division, we can use symbols and words.

Operation	Notation	Expression	Read as	Result
Division	$a \div b$ $\frac{a}{b}$ $b\overline{)a}$ a/b	$12 \div 4$ $\frac{12}{4}$ $4\overline{)12}$ $12/4$	Twelve divided by four	the quotient of 12 and 4

Division is performed on two numbers at a time. When translating from math notation to English words, or English words to math notation, look for the words *of* and *and* to identify the numbers.

Example:
Exercise:

Problem: Translate from math notation to words.

ⓐ $64 \div 8$ ⓑ $\frac{42}{7}$ ⓒ $4\overline{)28}$ ⓓ $64/8$

Solution:
Solution

- ⓐ We read this as *sixty-four divided by eight* and the result is *the quotient of sixty-four and eight*.
- ⓑ We read this as *forty-two divided by seven* and the result is *the quotient of forty-two and seven*.
- ⓒ We read this as *twenty-eight divided by four* and the result is *the quotient of twenty-eight and four*.
- ⓓ We read this as *sixty-four divided by eight* and the result is *the quotient of sixty-four and eight*.

Note:
Exercise:

Problem: Translate from math notation to words:

ⓐ $84 \div 7$ ⓑ $\frac{18}{6}$ ⓒ $8\overline{)24}$ ⓓ $18/6$

Solution:

- ⓐ eighty-four divided by seven; the quotient of eighty-four and seven
- ⓑ eighteen divided by six; the quotient of eighteen and six.
- ⓒ twenty-four divided by eight; the quotient of twenty-four and eight
- ⓓ eighteen divided by six; the quotient of eighteen and six.

Note:
Exercise:

Problem: Translate from math notation to words:

ⓐ $72 \div 9$ ⓑ $\frac{21}{3}$ ⓒ $6\overline{)54}$ ⓓ $72/9$

Solution:

- ⓐ seventy-two divided by nine; the quotient of seventy-two and nine
- ⓑ twenty-one divided by three; the quotient of twenty-one and three
- ⓒ fifty-four divided by six; the quotient of fifty-four and six
- ⓓ seventy-two divided by nine; the quotient of seventy-two and nine

Model Division of Whole Numbers

As we did with multiplication, we will model division using counters. The operation of division helps us organize items into equal groups as we start with the number of items in the dividend and subtract the number in the divisor repeatedly.

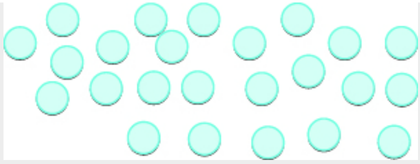
Example:
Exercise:

Problem: Model the division: $24 \div 8$.

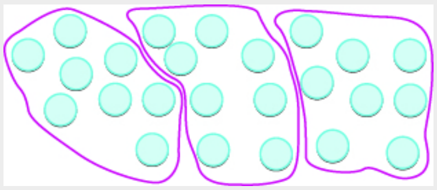
Solution:
Solution

To find the quotient $24 \div 8$, we want to know how many groups of 8 are in 24.

Model the dividend. Start with 24 counters.



The divisor tell us the number of counters we want in each group. Form groups of 8 counters.



Count the number of groups. There are 3 groups.

$24 \div 8 = 3$

Division as Repeated Subtraction
We can think of division as repeated subtraction where we are trying to find out how many times the divisor can be subtracted from the dividend.
$24 - 8 = 16$
$16 - 8 = 8$
$8 - 8 = 0$
And the quotient (answer) is 3 because 8 could be subtracted 3 times from 24.
Division and Multiplication are Opposites
Addition is the opposite of subtraction. Repeated subtraction can be undone by repeated addition.
$0 + 8 = 8$
$8 + 8 = 16$
$16 + 8 = 24$

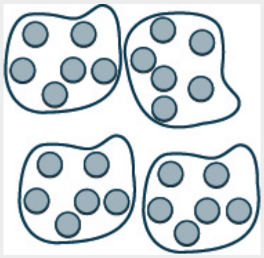
Repeated addition can be replaced by multiplication. That makes division and multiplication opposites.
$24 \div 8 = 3$
$3 \cdot 8 = 24$
Another way to think of Division
Another way to think of dividing 24 by 8 is to think of having 24 counters and splitting it evenly for 8 people. The question is how many counters will each person get. Here we want to know the size of the portion rather than how many portions of a particular size can we make.

Before we were thinking, how many groups of 8 counters can we make with 24 counters: a x 8 = 24
Now we are thinking, how big will the groups be if I make 8 groups using 24 counters: 8 x a = 24.
The commutative property of multiplication tells us that the order of the factors doesn't matter, so we can interpret division either way.
In both cases a = 3.

Note:
Exercise:

Problem: Model: $24 \div 6$.

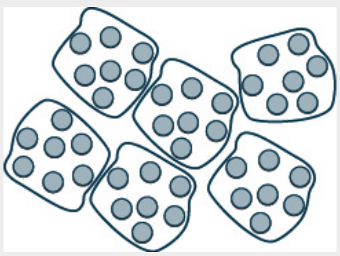
Solution:



Note:
Exercise:

Problem: Model: $42 \div 7$.

Solution:

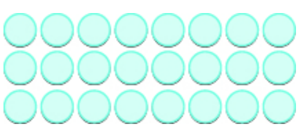


Divide Whole Numbers

We said that addition and subtraction are inverse operations because one undoes the other. Similarly, division is the inverse operation of multiplication. We know $12 \div 4 = 3$ because $3 \cdot 4 = 12$. Knowing all the multiplication number facts is very helpful when doing division.

We check our answer to division by multiplying the quotient by the divisor to determine if it equals the dividend. In [link], we know $24 \div 8 = 3$ is correct because $3 \cdot 8 = 24$.

If we arrange the counters of each group in a row and put the rows one on top of the other, then we can better see the relationship to multiplication.



This gives us a related way to understand division. We can think of division as making a rectangle where we know the area and the length of one side and we want to find the length of the side we do not yet know.

Example:
Exercise:

Problem: Divide. Then check by multiplying. ⓐ $42 \div 6$ ⓑ $\frac{72}{9}$ ⓒ $7\overline{)63}$

Solution:
Solution

ⓐ	
	mtd
Divide 42 by 6.	7
Check by multiplying. $7 \cdot 6$	
$42\checkmark$	

ⓑ	
	$\frac{72}{9}$
Divide 72 by 9.	8
Check by multiplying. $8 \cdot 9$	
$72\checkmark$	

ⓒ	
	$7\overline{)63}$

Divide 63 by 7.	9
Check by multiplying. $9 \cdot 7$	
$63\checkmark$	

Note:
Exercise:

Problem: Divide. Then check by multiplying:

ⓐ $54 \div 6$ ⓑ $\frac{27}{9}$

Solution:

ⓐ 9 ⓑ 3

Note:
Exercise:

Problem: Divide. Then check by multiplying:

ⓐ $\frac{36}{9}$ ⓑ $8\overline{)40}$

Solution:

ⓐ 4 ⓑ 5

What is the quotient when you divide a number by itself?
Equation:

$$\frac{15}{15} = 1 \text{ because } 1 \cdot 15 = 15$$

Dividing any number $(\text{except } 0)$ by itself produces a quotient of 1. Also, any number divided by 1 produces a quotient of that number. These two ideas are stated in the Division Properties of One.

Note:
Division Properties of One

Any number (except 0) divided by itself is one.	$a \div a = 1$
Any number divided by one is the same number.	$a \div 1 = \mathrm{a}$

Example:
Exercise:

Problem: Divide. Then check by multiplying:

ⓐ $11 \div 11$
ⓑ $\frac{19}{1}$
ⓒ $1\overline{)7}$

Solution:
Solution

ⓐ	
	$11 \div 11$
A number divided by itself is 1.	1
Check by multiplying. $1 \cdot 11$	
11✓	

ⓑ	
	$\frac{19}{1}$
A number divided by 1 equals itself.	19
Check by multiplying. $19 \cdot 1$	
19✓	

-

ⓒ	
	$1\overline{)7}$
A number divided by 1 equals itself.	7
Check by multiplying. $7 \cdot 1$	
7✓	

Note:
Exercise:

Problem: Divide. Then check by multiplying:

ⓐ $14 \div 14$ ⓑ $\frac{27}{1}$

Solution:

ⓐ 1
ⓑ 27

Note:
Exercise:

Problem: Divide. Then check by multiplying:

ⓐ $\frac{16}{1}$ ⓑ $1\overline{)4}$

Solution:

ⓐ 16
ⓑ 4

Note:
Division Properties of Zero
Zero divided by any number other than zero is zero
Suppose we have $0, and want to divide it among 3 people. How much would each person get? Each person would get $0. Zero divided by any number other than zero is 0.

We can also understand this using the area model of division.
We want to know the length of the other side of a rectangle if one side is 3 units and the area is 0 square units.

Any length other than 0 units would result in an area greater than 0 square units but 3 x 0 = 0 so the length must be 0 units.
To use the area model, we think of the 3 units by 0 units as a "degenerate" rectangle. Here that means one of the dimensions has shrunk to 0.
Division by zero is undefined
Now suppose that we want to divide $10 by 0people.
Thought of as repeated subtraction, this is asking how many groups can I form if I give each group $0. When I subtract $0 from $10 I still have $10 so this process never ends so there is no answer.

We can also understand this using the area model of division.
We want to know the length of the other side of a rectangle if one side is 0 units and the area is 10 square units.
Any length times 0 units results in an area of 0 square units so there is no length that gives 10 square units.

Dividing zero by zero also makes no sense and is also undefined. Thought of as multiplication it is 0 x $a = 0$. This is true for any value of a and thus there is no unique answer.

This is why dividing any number by 0 is undefined.

Zero divided by any number other than zero is 0.	$0 \div a = 0$
Dividing a number by zero is undefined.	$a \div 0$ undefined

Example:
Exercise:

Problem: Divide. Check by multiplying: ⓐ $0 \div 3$ ⓑ $10/0$.

Solution:
Solution

-
ⓐ	
	$0 \div 3$
Zero divided by any number is zero.	0
Check by multiplying. $0 \cdot 3$	
0✓	

-

ⓑ	
	$10/0$
Division by zero is undefined.	undefined

Note:
Exercise:

Problem: Divide. Then check by multiplying:

ⓐ $0 \div 2$ ⓑ $17/0$

Solution:

ⓐ 0 ⓑ undefined

Note:
Exercise:

Problem: Divide. Then check by multiplying:

ⓐ $0 \div 6$ ⓑ $13/0$

Solution:

ⓐ 0 ⓑ undefined

No Commutative Property of Division
There is not a commutative property for division. All it takes is one counterexamples to show this. Consider if 2/1 = 1/2. 2/1 = 2 while 1/2 is one-half. We'll learn more about fractions later, but for now I think everyone will agree that $2 is not the same as half a dollar.

No Associative Property of Division
There is not an associative property for division. Consider (8/4)/2 versus 8/(4/2). 8/4 = 2 and 2/2 = 1. While 4/2 = 2 and 8/2 = 4. This shows that there is no associative property for division because 1 is not equal to 4.

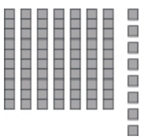
Distributive Property of Division
This property is (a + b)/c = a/c + b/c. Frequently division involves fractions so we'll look at this in more detail when we learn fractions. Here is an example that avoids fractions. (18+6)/3 = 18/3 + 6/3. 18+6 = 24 and 24/3 =

8. While 18/3 = 6 and 6/3 = 2, and 6+2=8. Thinking of counters, this says that for two piles of counters that to be divide up evenly, one can add them together first and then divide them or one can divide them separately and then add those results together. The result is the same.

Another way to think of this is with money. Imagine you and a friend did two small jobs together but you did most of the work. So instead of splitting the money evenly, both of you agreed that your friend would only get one-third of the pay. One job paid $18 and the other paid $6. One way of figuring your friends share is ($18+$6)/3 = $24/3 = $8. In other words add the two amounts together and then give your friend one-third. The other way is to give your friend one-third of each job and then add those together: $18/3 + $6/3 = $6 + $2 = $8. Either way is correct and your friend gets $8.

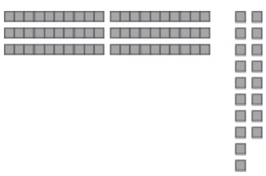
Here is a larger example using base-10 blocks and the Area Model of Multiplication.
78 divided by 3 can be thought of as computing the number that when multiplied by 3 equals 78.
Thought of as a rectangle, one side is 3 and the area is 78. What does the other side have to be?
One way to solve this problem is to get 78 as base-10 blocks, and using all of the blocks make a single rectangle where one of the sides is 3.
The other side will be the solution to the division problem.

We could get 78 units but that's a lot of work and doesn't allow us to take advantage of place value. Instead, start with 7 rods and 8 units, the normal way to represent 78.

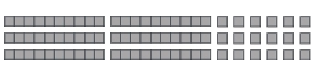


Make 3 equal rows using as many rods as possible. That's 2 rods per row. There is one rod left over along with the unused 8 units.

Exchange the remaining rod for 10 units. Combine that with the 8 units you already had giving you 18 units.



Now divide those 18 units equally over the 3 rows. Each row gets 6 units so all 18 are used and no base-10 blocks remain not in a row. Notice that each row has a value of 26, two rods and 6 units. 26 must be the answer to the original division problem. Check that 3 x 26 = 78.



This used the Distributive Property of Division because it broke up dividing 78 by 3 into dividing 70 by 3 and then 8 by 3.
70 was then split up into 60 + 10 using the property a second time.
After the 60/3 was calculated, the 10 (the one remaining rod) had to converted into units and put with other units.
This used the Distributive Property of Division in the other direction. Finally the 18 was divided by 3.

When the divisor or the dividend has more than one digit, it is usually easier to use the $4\overline{)12}$ notation when working on paper. This process is called long division. Let’s work through the process by dividing 78 by 3. You should see how it directly follows what we just did with base-10 blocks.

Divide the first digit of the dividend, 7, by the divisor, 3.	
The divisor 3 can go into 7 two times since $2 \times 3 = 6$. Write the 2 above the 7 in the quotient.	2 3)78
Multiply the 2 in the quotient by 2 and write the product, 6, under the 7.	2 3)78 6
Subtract that product from the first digit in the dividend. Subtract $7 - 6$. Write the difference, 1, under the first digit in the dividend.	2 3)78 6 1
Bring down the next digit of the dividend. Bring down the 8.	2 3)78 6 18
Divide 18 by the divisor, 3. The divisor 3 goes into 18 six times.	26 3)78 6 18
Write 6 in the quotient above the 8.	
Multiply the 6 in the quotient by the divisor and write the product, 18, under the dividend. Subtract 18 from 18.	26 3)78 6 18 18 0

We would repeat the process until there are no more digits in the dividend to bring down. In this problem, there are no more digits to bring down, so the division is finished.

Equation:

$$\text{So } 78 \div 3 = 26.$$

Money Model

For practice, do this same problem with money. Start with 7 \$10 bills and 8 \$1 bills. Divide the \$78 into three equal piles.

As you are dividing the money, try to follow the long division steps shown earlier.

Check Division with Multiplication

Check by multiplying the quotient times the divisor to get the dividend. Multiply 26×3 to make sure that product equals the dividend, 78.

Equation:

$$\begin{array}{r} \overset{1}{2}6 \\ \underline{\times 3} \\ 78\checkmark \end{array}$$

It does, so our answer is correct.

Note:
Divide whole numbers.

Divide the first digit of the dividend by the divisor. If the divisor is larger than the first digit of the dividend, divide the first two digits of the dividend by the divisor, and so on.
Write the quotient above the dividend.
Multiply the quotient by the divisor and write the product under the dividend.
Subtract that product from the dividend.
Bring down the next digit of the dividend.
Repeat from Step 1 until there are no more digits in the dividend to bring down.
Check by multiplying the quotient times the divisor.

Example:
Exercise:

Problem: Divide $2{,}596 \div 4$. Check by multiplying:

Solution:
Solution

Let's rewrite the problem to set it up for long division.	$4\overline{)2596}$
Divide the first digit of the dividend, 2, by the divisor, 4.	$4\overline{)2596}$
Since 4 does not go into 2, we use the first two digits of the dividend and divide 25 by 4. The divisor 4 goes into 25 six times.	
We write the 6 in the quotient above the 5.	$\begin{array}{r} 6 \\ 4\overline{)2596} \end{array}$
Multiply the 6 in the quotient by the divisor 4 and write the product, 24, under the first two digits in the dividend.	$\begin{array}{r} 6 \\ 4\overline{)2596} \\ \underline{24} \end{array}$

Subtract that product from the first two digits in the dividend. Subtract $25-24$. Write the difference, 1, under the second digit in the dividend.	$\begin{array}{r} 6 \\ 4\overline{)2596} \\ 24 \\ \hline 1 \end{array}$
Now bring down the 9 and repeat these steps. There are 4 fours in 19. Write the 4 over the 9. Multiply the 4 by 4 and subtract this product from 19.	$\begin{array}{r} 64 \\ 4\overline{)2596} \\ 24 \\ \hline 19 \\ 16 \\ \hline 3 \end{array}$
Bring down the 6 and repeat these steps. There are 9 fours in 36. Write the 9 over the 6. Multiply the 9 by 4 and subtract this product from 36.	$\begin{array}{r} 649 \\ 4\overline{)2596} \\ 24 \\ \hline 19 \\ 16 \\ \hline 36 \\ 36 \\ \hline 0 \end{array}$
So $2{,}596 \div 4 = 649$.	
Check by multiplying. $\begin{array}{r} \overset{1\,3}{649} \\ \times \quad 4 \\ \hline 2{,}596 \checkmark \end{array}$	

It equals the dividend, so our answer is correct.

Note:
Exercise:

Problem: Divide. Then check by multiplying: $2{,}636 \div 4$

Solution:

659

Note:
Exercise:

Problem: Divide. Then check by multiplying: $2{,}716 \div 4$

Solution:

679

Example:
Exercise:

Problem: Divide 4,506 ÷ 6. Check by multiplying:

Solution:
Solution

Let's rewrite the problem to set it up for long division.	$6\overline{)4506}$
First we try to divide 6 into 4.	$6\overline{)4506}$
Since that won't work, we try 6 into 45. There are 7 sixes in 45. We write the 7 over the 5.	$\begin{array}{r} 7 \\ 6\overline{)4506} \end{array}$
Multiply the 7 by 6 and subtract this product from 45.	$\begin{array}{r} 7 \\ 6\overline{)4506} \\ \underline{42} \\ 3 \end{array}$
Now bring down the 0 and repeat these steps. There are 5 sixes in 30. Write the 5 over the 0. Multiply the 5 by 6 and subtract this product from 30.	$\begin{array}{r} 75 \\ 6\overline{)4506} \\ \underline{42} \\ 30 \\ \underline{30} \\ 0 \end{array}$
Now bring down the 6 and repeat these steps. There is 1 six in 6. Write the 1 over the 6. Multiply 1 by 6 and subtract this product from 6.	$\begin{array}{r} 751 \\ 6\overline{)4506} \\ \underline{42} \\ 30 \\ \underline{30} \\ 06 \\ \underline{6} \\ 0 \end{array}$
Check by multiplying. $\begin{array}{r} \overset{3}{7}51 \\ \times\quad 6 \\ \hline 4{,}506 \end{array}$ ✓	

It equals the dividend, so our answer is correct.

Note:
Exercise:

Problem: Divide. Then check by multiplying: $4,305 \div 5$.

Solution:

861

Note:
Exercise:

Problem: Divide. Then check by multiplying: $3,906 \div 6$.

Solution:

651

Example:
Exercise:

Problem: Divide $7,263 \div 9$. Check by multiplying.

Solution:
Solution

Let's rewrite the problem to set it up for long division.	$9\overline{)7263}$
First we try to divide 9 into 7.	$9\overline{)7263}$
Since that won't work, we try 9 into 72. There are 8 nines in 72. We write the 8 over the 2.	$\begin{array}{r} 8 \\ 9\overline{)7263} \end{array}$

Multiply the 8 by 9 and subtract this product from 72.	$\begin{array}{r} 8 \\ 9\overline{)7263} \\ \underline{72} \\ 0 \end{array}$
Now bring down the 6 and repeat these steps. There are 0 nines in 6. Write the 0 over the 6. Multiply the 0 by 9 and subtract this product from 6.	$\begin{array}{r} 80 \\ 9\overline{)7263} \\ \underline{72} \\ 06 \\ \underline{0} \\ 6 \end{array}$
Now bring down the 3 and repeat these steps. There are 7 nines in 63. Write the 7 over the 3. Multiply the 7 by 9 and subtract this product from 63.	$\begin{array}{r} 807 \\ 9\overline{)7263} \\ \underline{72} \\ 06 \\ \underline{0} \\ 63 \\ \underline{63} \\ 0 \end{array}$
Check by multiplying. $\begin{array}{r} \overset{6}{8}07 \\ \underline{\times \quad 9} \\ 7{,}263 \checkmark \end{array}$	

It equals the dividend, so our answer is correct.

Note:
Exercise:

Problem: Divide. Then check by multiplying: 4,928 ÷ 7.

Solution:

704

Note:
Exercise:

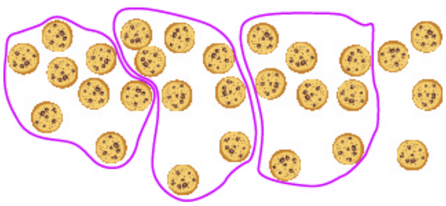
Problem: Divide. Then check by multiplying: 5,663 ÷ 7.

Solution:

809

So far all the division problems have worked out evenly. For example, if we had 24 cookies and wanted to make bags of 8 cookies, we would have 3 bags. But what if there were 28 cookies and we wanted to make bags of 8? Start with the 28 cookies as shown in [link].

Try to put the cookies in groups of eight as in [link].



There are 3 groups of eight cookies, and 4 cookies left over. We call the 4 cookies that are left over the remainder and show it by writing R4 next to the 3. (The R stands for remainder.)

To check this division we multiply 3 times 8 to get 24, and then add the remainder of 4.
Equation:

$$\begin{array}{r} 3 \\ \underline{\times 8} \\ 24 \\ \underline{+4} \\ 28 \end{array}$$

Example:
Exercise:

Problem: Divide $1{,}439 \div 4$. Check by multiplying.

Solution:
Solution

Let's rewrite the problem to set it up for long division.	$4\overline{)1439}$

First we try to divide 4 into 1. Since that won't work, we try 4 into 14. There are 3 fours in 14. We write the 3 over the 4.	$\begin{array}{r} 3 \\ 4\overline{)1439} \end{array}$
Multiply the 3 by 4 and subtract this product from 14.	$\begin{array}{r} 3 \\ 4\overline{)1439} \\ \underline{12} \\ 2 \end{array}$
Now bring down the 3 and repeat these steps. There are 5 fours in 23. Write the 5 over the 3. Multiply the 5 by 4 and subtract this product from 23.	$\begin{array}{r} 35 \\ 4\overline{)1439} \\ \underline{12}\downarrow \\ 23 \\ \underline{20} \\ 3 \end{array}$
Now bring down the 9 and repeat these steps. There are 9 fours in 39. Write the 9 over the 9. Multiply the 9 by 4 and subtract this product from 39. There are no more numbers to bring down, so we are done. The remainder is 3.	$\begin{array}{r} 359\text{R}3 \\ 4\overline{)1439} \\ \underline{12} \\ 23 \\ \underline{20}\downarrow \\ 39 \\ \underline{36} \\ 3 \end{array}$
Check by multiplying. $\begin{array}{rl} \overset{2\,3}{359} & \text{quotient} \\ \times\quad 4 & \text{divisor} \\ \hline 1{,}436 & \\ +\quad 3 & \text{remainder} \\ \hline 1{,}439\checkmark & \end{array}$	

So 1,439 ÷ 4 is 359 with a remainder of 3. Our answer is correct.

Note:
Exercise:

Problem: Divide. Then check by multiplying: 3,812 ÷ 8.

Solution:

476 with a remainder of 4

Note:
Exercise:

Problem: Divide. Then check by multiplying: $4,319 \div 8$.

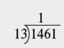
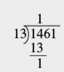
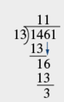
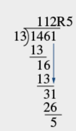
Solution:

539 with a remainder of 7

Example:
Exercise:

Problem: Divide and then check by multiplying: $1,461 \div 13$.

Solution:
Solution

Let's rewrite the problem to set it up for long division.	$13\overline{)1,461}$
First we try to divide 13 into 1. Since that won't work, we try 13 into 14. There is 1 thirteen in 14. We write the 1 over the 4.	
Multiply the 1 by 13 and subtract this product from 14.	
Now bring down the 6 and repeat these steps. There is 1 thirteen in 16. Write the 1 over the 6. Multiply the 1 by 13 and subtract this product from 16.	
Now bring down the 1 and repeat these steps. There are 2 thirteens in 31. Write the 2 over the 1. Multiply the 2 by 13 and subtract this product from 31. There are no more numbers to bring down, so we are done. The remainder is 5. $1,462 \div 13$ is 112 with a remainder of 5.	
Check by multiplying.	

$$\begin{array}{rl} 112 & \text{quotient} \\ \times\ 13 & \text{divisor} \\ \hline 336 & \\ 1{,}120 & \\ +\ \ 5 & \text{remainder} \\ \hline 1{,}461 & \checkmark \end{array}$$	

Our answer is correct.

Note:
Exercise:

Problem: Divide. Then check by multiplying: $1{,}493 \div 13$.

Solution:

114 R11

Note:
Exercise:

Problem: Divide. Then check by multiplying: $1{,}461 \div 12$.

Solution:

121 R9

Example:
Exercise:

Problem: Divide and check by multiplying: $74{,}521 \div 241$.

Solution:
Solution

Let's rewrite the problem to set it up for long division.	$241\overline{)74{,}521}$
First we try to divide 241 into 7. Since that won't work, we try 241 into 74. That still won't work, so we try 241 into 745. Since 2 divides into 7 three times, we try 3.	

Since $3 \times 241 = 723$, we write the 3 over the 5 in 745. Note that 4 would be too large because $4 \times 241 = 964$, which is greater than 745.	
Multiply the 3 by 241 and subtract this product from 745.	$\begin{array}{r} 3 \\ 241\overline{)74521} \\ \underline{723} \\ 22 \end{array}$
Now bring down the 2 and repeat these steps. 241 does not divide into 222. We write a 0 over the 2 as a placeholder and then continue.	$\begin{array}{r} 3 \\ 241\overline{)74521} \\ \underline{723} \\ 22 \end{array}$
Now bring down the 1 and repeat these steps. Try 9. Since $9 \times 241 = 2{,}169$, we write the 9 over the 1. Multiply the 9 by 241 and subtract this product from 2,221.	$\begin{array}{r} 309 \text{ R52} \\ 241\overline{)74521} \\ \underline{723} \\ 2221 \\ \underline{2169} \\ 52 \end{array}$
There are no more numbers to bring down, so we are finished. The remainder is 52. So $74{,}521 \div 241$ is 309 with a remainder of 52.	
Check by multiplying. $\begin{array}{rl} \overset{3}{309} & \text{quotient} \\ \underline{\times\ 241} & \text{divisor} \\ 309 & \\ 12{,}360 & \\ \underline{61{,}800} & \\ 72{,}469 & \\ \underline{+\quad 52} & \text{remainder} \\ 74{,}521 & \checkmark \end{array}$	

Sometimes it might not be obvious how many times the divisor goes into digits of the dividend. We will have to guess and check numbers to find the greatest number that goes into the digits without exceeding them.

Note:
Exercise:

Problem: Divide. Then check by multiplying: $78{,}641 \div 256$.

Solution:

307 R49

Note:
Exercise:

Problem: Divide. Then check by multiplying: $76{,}461 \div 248$.

Solution:

308 R77

Translate Word Phrases to Math Notation

Earlier in this section, we translated math notation for division into words. Now we'll translate word phrases into math notation. Some of the words that indicate division are given in [link].

Operation	Word Phrase	Example	Expression
Division	divided by quotient of divided into	12 divided by 4 the quotient of 12 and 4 4 divided into 12	$12 \div 4$ $\frac{12}{4}$ $12/4$ $4\overline{)12}$

Example:
Exercise:

Problem: Translate and simplify: the quotient of 51 and 17.

Solution:
Solution

The word *quotient* tells us to divide.

	the quotient of 51 and 17
Translate.	$51 \div 17$
Divide.	3

We could just as correctly have translated *the quotient of* 51 *and* 17 using the notation

$17\overline{)51}$ or $\frac{51}{17}$.

Note:
Exercise:

Problem: Translate and simplify: the quotient of 91 and 13.

Solution:

91 ÷ 13; 7

Note:
Exercise:

Problem: Translate and simplify: the quotient of 52 and 13.

Solution:

52 ÷ 13; 4

Divide Whole Numbers in Applications

We will use the same strategy we used in previous sections to solve applications.

- Determine what we are asked to find.
- Write a phrase that gives the information to find it.
- Translate the phrase into math notation.
- Simplify to get the answer.
- Write a sentence to answer the question, using the appropriate units.

Example:
Exercise:

Problem:

Cecelia bought a 160-ounce box of oatmeal at the big box store. She wants to divide the 160 ounces of oatmeal into 8-ounce servings. She will put each serving into a plastic bag so she can take one bag to work each day. How many servings will she get from the big box?

Solution:
Solution

We are asked to find the how many servings she will get from the big box.

Write a phrase.	160 ounces divided by 8 ounces
Translate to math notation.	$160 \div 8$

Simplify by dividing.	20
Write a sentence to answer the question.	Cecelia will get 20 servings from the big box.

Note:
Exercise:

Problem:

Marcus is setting out animal crackers for snacks at the preschool. He wants to put 9 crackers in each cup. One box of animal crackers contains 135 crackers. How many cups can he fill from one box of crackers?

Solution:

Marcus can fill 15 cups.

Note:
Exercise:

Problem:

Andrea is making bows for the girls in her dance class to wear at the recital. Each bow takes 4 feet of ribbon, and 36 feet of ribbon are on one spool. How many bows can Andrea make from one spool of ribbon?

Solution:

Andrea can make 9 bows.

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Dividing Whole Numbers
- Dividing Whole Numbers No Remainder
- Dividing Whole Numbers With Remainder

Key Concepts

Operation	Notation	Expression	Read as	Result

Operation	Notation	Expression	Read as	Result
Division	$\div$ $\frac{a}{b}$ $b\overline{)a}$ a/b	$12 \div 4$ $\frac{12}{4}$ $4\overline{)12}$ $12/4$	Twelve divided by four	the quotient of 12 and 4

- **Division Properties of One**
  - Any number (except 0) divided by itself is one. $a \div a = 1$
  - Any number divided by one is the same number. $a \div 1 = a$
- **Division Properties of Zero**
  - Zero divided by any number is 0. $0 \div a = 0$
  - Dividing a number by zero is undefined. $a \div 0$ undefined
- **Divide whole numbers.**

  Divide the first digit of the dividend by the divisor. If the divisor is larger than the first digit of the dividend, divide the first two digits of the dividend by the divisor, and so on.
  Write the quotient above the dividend.
  Multiply the quotient by the divisor and write the product under the dividend.
  Subtract that product from the dividend.
  Bring down the next digit of the dividend.
  Repeat from Step 1 until there are no more digits in the dividend to bring down.
  Check by multiplying the quotient times the divisor.

Practice Makes Perfect

Use Division Notation

In the following exercises, translate from math notation to words.
Exercise:

Problem: $54 \div 9$

Solution:

fifty-four divided by nine; the quotient of fifty-four and nine

Exercise:

Problem: $\frac{56}{7}$

Exercise:

Problem: $\frac{32}{8}$

Solution:

thirty-two divided by eight; the quotient of thirty-two and eight

Exercise:

Problem: $6\overline{)42}$

Exercise:

Problem: $48 \div 6$

Solution:

forty-eight divided by six; the quotient of forty-eight and six

Exercise:

Problem: $\frac{63}{9}$

Exercise:

Problem: $7\overline{)63}$

Solution:

sixty-three divided by seven; the quotient of sixty-three and seven

Exercise:

Problem: $72 \div 8$

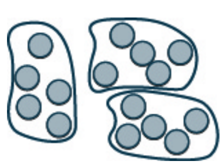
Model Division of Whole Numbers

In the following exercises, model the division.

Exercise:

Problem: $15 \div 5$

Solution:



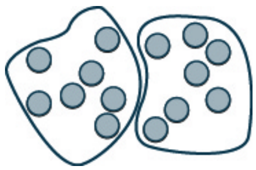
Exercise:

Problem: $10 \div 5$

Exercise:

Problem: $\frac{14}{7}$

Solution:



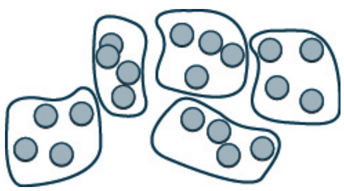
Exercise:

Problem: $\frac{18}{6}$

Exercise:

Problem: $4\overline{)20}$

Solution:



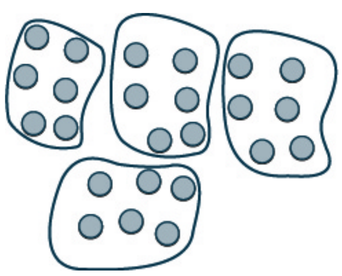
Exercise:

Problem: $3\overline{)15}$

Exercise:

Problem: $24 \div 6$

Solution:



Exercise:

Problem: $16 \div 4$

Divide Whole Numbers

In the following exercises, divide. Then check by multiplying.

Exercise:

Problem: $18 \div 2$

Solution:

9

Exercise:

Problem: $14 \div 2$

Exercise:

Problem: $\frac{27}{3}$

Solution:

9

Exercise:

Problem: $\frac{30}{3}$

Exercise:

Problem: $4\overline{)28}$

Solution:

7

Exercise:

Problem: $4\overline{)36}$

Exercise:

Problem: $\frac{45}{5}$

Solution:

9

Exercise:

Problem: $\frac{35}{5}$

Exercise:

Problem: $72/8$

Solution:

9

Exercise:

Problem: $8\overline{)64}$

Exercise:

Problem: $\frac{35}{7}$

Solution:

5

Exercise:

Problem: $42 \div 7$

Exercise:

Problem: $15\overline{)15}$

Solution:

1

Exercise:

Problem: $12\overline{)12}$

Exercise:

Problem: $43 \div 43$

Solution:

1

Exercise:

Problem: $37 \div 37$

Exercise:

Problem: $\frac{23}{1}$

Solution:

23

Exercise:

Problem: $\frac{29}{1}$

Exercise:

Problem: $19 \div 1$

Solution:

19

Exercise:

Problem: $17 \div 1$

Exercise:

Problem: $0 \div 4$

Solution:

0

Exercise:

Problem: $0 \div 8$

Exercise:

Problem: $\frac{5}{0}$

Solution:

undefined

Exercise:

Problem: $\frac{9}{0}$

Exercise:

Problem: $\frac{26}{0}$

Solution:

undefined

Exercise:

Problem: $\frac{32}{0}$

Exercise:

Problem: $12\overline{)0}$

Solution:

0

Exercise:

Problem: $16\overline{)0}$

Exercise:

Problem: $72 \div 3$

Solution:

24

Exercise:

Problem: $57 \div 3$

Exercise:

Problem: $\frac{96}{8}$

Solution:

12

Exercise:

Problem: $\frac{78}{6}$

Exercise:

Problem: $5\overline{)465}$

Solution:

93

Exercise:

Problem: $4\overline{)528}$

Exercise:

Problem: $924 \div 7$

Solution:

132

Exercise:

Problem: $861 \div 7$

Exercise:

Problem: $\frac{5{,}226}{6}$

Solution:

871

Exercise:

Problem: $\frac{3,776}{8}$

Exercise:

Problem: $4\overline{)31,324}$

Solution:

7,831

Exercise:

Problem: $5\overline{)46,855}$

Exercise:

Problem: $7,209 \div 3$

Solution:

2,403

Exercise:

Problem: $4,806 \div 3$

Exercise:

Problem: $5,406 \div 6$

Solution:

901

Exercise:

Problem: $3,208 \div 4$

Exercise:

Problem: $4\overline{)2,816}$

Solution:

704

Exercise:

Problem: $6\overline{)3,624}$

Exercise:

Problem: $\frac{91,881}{9}$

Solution:

10,209

Exercise:

Problem: $\frac{83,256}{8}$

Exercise:

Problem: $2,470 \div 7$

Solution:

352 R6

Exercise:

Problem: $3,741 \div 7$

Exercise:

Problem: $8\overline{)55,305}$

Solution:

6,913 R1

Exercise:

Problem: $9\overline{)51,492}$

Exercise:

Problem: $\frac{431,174}{5}$

Solution:

86,234 R4

Exercise:

Problem: $\frac{297,277}{4}$

Exercise:

Problem: $130,016 \div 3$

Solution:

43,338 R2

Exercise:

Problem: $105{,}609 \div 2$

Exercise:

Problem: $15\overline{)5{,}735}$

Solution:

382 R5

Exercise:

Problem: $\frac{4{,}933}{21}$

Exercise:

Problem: $56{,}883 \div 67$

Solution:

849

Exercise:

Problem: $43{,}725/75$

Exercise:

Problem: $\frac{30{,}144}{314}$

Solution:

96

Exercise:

Problem: $26{,}145 \div 415$

Exercise:

Problem: $273\overline{)542{,}195}$

Solution:

1,986 R17

Exercise:

Problem: $816{,}243 \div 462$

Mixed Practice

In the following exercises, simplify.
Exercise:

Problem: $15\,(204)$

Solution:

3,060

Exercise:

Problem: $74 \cdot 391$

Exercise:

Problem: $256 - 184$

Solution:

72

Exercise:

Problem: $305 - 262$

Exercise:

Problem: $719 + 341$

Solution:

1,060

Exercise:

Problem: $647 + 528$

Exercise:

Problem: $25\overline{)875}$

Solution:

35

Exercise:

Problem: $1104 \div 23$

Translate Word Phrases to Algebraic Expressions

In the following exercises, translate and simplify.
Exercise:

Problem: the quotient of 45 and 15

Solution:

$45 \div 15; 3$

Exercise:

Problem: the quotient of 64 and 16

Exercise:

Problem: the quotient of 288 and 24

Solution:

$288 \div 24; 12$

Exercise:

Problem: the quotient of 256 and 32

Divide Whole Numbers in Applications

In the following exercises, solve.

Exercise:

Problem:

Trail mix Ric bought 64 ounces of trail mix. He wants to divide it into small bags, with 2 ounces of trail mix in each bag. How many bags can Ric fill?

Solution:

Ric can fill 32 bags.

Exercise:

Problem:

Crackers Evie bought a 42 ounce box of crackers. She wants to divide it into bags with 3 ounces of crackers in each bag. How many bags can Evie fill?

Exercise:

Problem:

Astronomy class There are 125 students in an astronomy class. The professor assigns them into groups of 5. How many groups of students are there?

Solution:

There are 25 groups.

Exercise:

Problem:

Flower shop Melissa's flower shop got a shipment of 152 roses. She wants to make bouquets of 8 roses each. How many bouquets can Melissa make?

Exercise:

Problem:

Baking One roll of plastic wrap is 48 feet long. Marta uses 3 feet of plastic wrap to wrap each cake she bakes. How many cakes can she wrap from one roll?

Solution:

Marta can wrap 16 cakes from 1 roll.

Exercise:

Problem:

Dental floss One package of dental floss is 54 feet long. Brian uses 2 feet of dental floss every day. How many days will one package of dental floss last Brian?

Mixed Practice

In the following exercises, solve.

Exercise:

Problem:

Miles per gallon Susana's hybrid car gets 45 miles per gallon. Her son's truck gets 17 miles per gallon. What is the difference in miles per gallon between Susana's car and her son's truck?

Solution:

The difference is 28 miles per gallon.

Exercise:

Problem:

Distance Mayra lives 53 miles from her mother's house and 71 miles from her mother-in-law's house. How much farther is Mayra from her mother-in-law's house than from her mother's house?

Exercise:

Problem:

Field trip The 45 students in a Geology class will go on a field trip, using the college's vans. Each van can hold 9 students. How many vans will they need for the field trip?

Solution:

They will need 5 vans for the field trip

Exercise:

Problem:

Potting soil Aki bought a 128 ounce bag of potting soil. How many 4 ounce pots can he fill from the bag?

Exercise:

Problem:

Hiking Bill hiked 8 miles on the first day of his backpacking trip, 14 miles the second day, 11 miles the third day, and 17 miles the fourth day. What is the total number of miles Bill hiked?

Solution:

Bill hiked 50 miles

Exercise:

Problem:

Reading Last night Emily read 6 pages in her Business textbook, 26 pages in her History text, 15 pages in her Psychology text, and 9 pages in her math text. What is the total number of pages Emily read?

Exercise:

Problem:

Patients LaVonne treats 12 patients each day in her dental office. Last week she worked 4 days. How many patients did she treat last week?

Solution:

LaVonne treated 48 patients last week.

Exercise:

Problem:

Scouts There are 14 boys in Dave's scout troop. At summer camp, each boy earned 5 merit badges. What was the total number of merit badges earned by Dave's scout troop at summer camp?

Writing Exercises

Exercise:

Problem: Explain how you use the multiplication facts to help with division.

Solution:

Answers may vary. Using multiplication facts can help you check your answers once you've finished division.

Exercise:

Problem:

Oswaldo divided 300 by 8 and said his answer was 37 with a remainder of 4. How can you check to make sure he is correct?

Everyday Math

Exercise:

Problem:

Contact lenses Jenna puts in a new pair of contact lenses every 14 days. How many pairs of contact lenses does she need for 365 days?

Solution:

Jenna uses 26 pairs of contact lenses, but there is 1 day left over, so she needs 27 pairs for 365 days.

Exercise:

Problem:

Cat food One bag of cat food feeds Lara's cat for 25 days. How many bags of cat food does Lara need for 365 days?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use division notation.			
model division of whole numbers.			
divide whole numbers.			
translate word phrases to algebraic expressions.			
divide whole numbers in applications.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next Chapter? Why or why not?

Chapter Review Exercises

Introduction to Whole Numbers

Identify Counting Numbers and Whole Numbers

In the following exercises, determine which of the following are (a) counting numbers (b) whole numbers.

Exercise:

Problem: 0, 2, 99

Solution:

ⓐ 2, 99
ⓑ 0, 2, 99

Exercise:

Problem: 0, 3, 25

Exercise:

Problem: 0, 4, 90

Solution:

ⓐ 4, 90
ⓑ 0, 4, 90

Exercise:

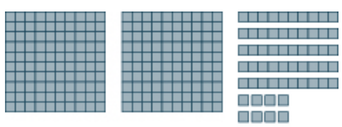
Problem: 0, 1, 75

Model Whole Numbers

In the following exercises, model each number using base-10 blocks and then show its value using place value notation.
Exercise:

Problem: 258

Solution:



Place Value	Digit	Total Value
hundreds	2	200
tens	5	50
ones	8	8
		258

Exercise:

Problem: 104

Identify the Place Value of a Digit

In the following exercises, find the place value of the given digits.
Exercise:

Problem: 472,981

ⓐ 8
ⓑ 4
ⓒ 1
ⓓ 7
ⓔ 2

Solution:

ⓐ tens
ⓑ hundred thousands
ⓒ ones

ⓓ thousands
ⓔ ten thousands

Exercise:

Problem: 12,403,295

ⓐ 4
ⓑ 0
ⓒ 1
ⓓ 9
ⓔ 3

Use Place Value to Name Whole Numbers

In the following exercises, name each number in words.

Exercise:

Problem: 5,280

Solution:

Five thousand two hundred eighty

Exercise:

Problem: 204,614

Exercise:

Problem: 5,012,582

Solution:

Five million twelve thousand five hundred eighty-two

Exercise:

Problem: 31,640,976

Use Place Value to Write Whole Numbers

In the following exercises, write as a whole number using digits.

Exercise:

Problem: six hundred two

Exercise:

Problem: fifteen thousand, two hundred fifty-three

Solution:

15,253

Exercise:

Problem: three hundred forty million, nine hundred twelve thousand, sixty-one

---

Solution:

340,912,061

Exercise:

Problem: two billion, four hundred ninety-two million, seven hundred eleven thousand, two

Round Whole Numbers

In the following exercises, round to the nearest ten.

Exercise:

Problem: 412

Solution:

410

Exercise:

Problem: 648

Exercise:

Problem: 3,556

Solution:

3,560

Exercise:

Problem: 2,734

In the following exercises, round to the nearest hundred.

Exercise:

Problem: 38,975

Solution:

39,000

Exercise:

Problem: 26,849

Exercise:

Problem: 81,486

Solution:

81,500

Exercise:

Problem: 75,992

Add Whole Numbers

Use Addition Notation

In the following exercises, translate the following from math notation to words.

Exercise:

Problem: $4 + 3$

Solution:

four plus three; the sum of four and three

Exercise:

Problem: $25 + 18$

Exercise:

Problem: $571 + 629$

Solution:

five hundred seventy-one plus six hundred twenty-nine; the sum of five hundred seventy-one and six hundred twenty-nine

Exercise:

Problem: $10,085 + 3,492$

Model Addition of Whole Numbers

In the following exercises, model the addition.

Exercise:

Problem: $6 + 7$

Solution:

Exercise:

Problem: $38 + 14$

Add Whole Numbers

In the following exercises, fill in the missing values in each chart.

Exercise:

Problem:

+	0	1	2	3	4	5	6	7	8	9
0	0	1		3	4		6	7		9
1	1	2	3	4			7	8	9	
2		3	4	5	6	7	8		10	11
3	3		5		7	8		10		12
4	4	5			8	9			12	
5	5		7	8			11		13	
6	6	7	8		10			13		15
7			9			12	13		15	16
8	8	9		11			14		16	
9	9	10	11		13	14			17	

Solution:

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Exercise:

Problem:

+	3	4	5	6	7	8	9
6							
7							
8							
9							

In the following exercises, add.

Exercise:

Problem: ⓐ $0 + 19$ ⓑ $19 + 0$

Solution:

ⓐ 19
ⓑ 19

Exercise:

Problem: ⓐ $0 + 480$ ⓑ $480 + 0$

Exercise:

Problem: ⓐ $7 + 6$ ⓑ $6 + 7$

Solution:

ⓐ 13
ⓑ 13

Exercise:

Problem: ⓐ $23 + 18$ ⓑ $18 + 23$

Exercise:

Problem: $44 + 35$

Solution:

82

Exercise:

Problem: $63 + 29$

Exercise:

Problem: $96 + 58$

Solution:

154

Exercise:

Problem: $375 + 591$

Exercise:

Problem: $7{,}281 + 12{,}546$

Solution:

19,827

Exercise:

Problem: $5{,}280 + 16{,}324 + 9{,}731$

Translate Word Phrases to Math Notation

In the following exercises, translate each phrase into math notation and then simplify.

Exercise:

Problem: the sum of 30 and 12

Solution:

30 + 12; 42

Exercise:

Problem: 11 increased by 8

Exercise:

Problem: 25 more than 39

Solution:

39 + 25; 64

Exercise:

Problem: total of 15 and 50

Add Whole Numbers in Applications

In the following exercises, solve.

Exercise:

Problem:

Shopping for an interview Nathan bought a new shirt, tie, and slacks to wear to a job interview. The shirt cost $24, the tie cost $14, and the slacks cost $38. What was Nathan's total cost?

Solution:

$76

Exercise:

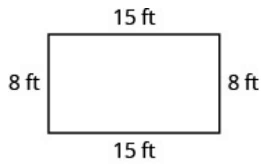
Problem:

Running Jackson ran 4 miles on Monday, 12 miles on Tuesday, 1 mile on Wednesday, 8 miles on Thursday, and 5 miles on Friday. What was the total number of miles Jackson ran?

In the following exercises, find the perimeter of each figure.

Exercise:

Problem:

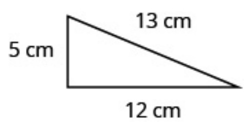


Solution:

46 feet

Exercise:

Problem:



[Subtract Whole Numbers]

Use Subtraction Notation

In the following exercises, translate the following from math notation to words.

Exercise:

Problem: $14 - 5$

Solution:

fourteen minus five; the difference of fourteen and five

Exercise:

Problem: $40 - 15$

Exercise:

Problem: $351 - 249$

Solution:

three hundred fifty-one minus two hundred forty-nine; the difference between three hundred fifty-one and two hundred forty-nine

Exercise:

Problem: $5{,}724 - 2{,}918$

Model Subtraction of Whole Numbers

In the following exercises, model the subtraction.

Exercise:

Problem: $18 - 4$

Solution:

Exercise:

Problem: $41 - 29$

Subtract Whole Numbers

In the following exercises, subtract and then check by adding.

Exercise:

Problem: $8 - 5$

Solution:

3

Exercise:

Problem: $12 - 7$

Exercise:

Problem: $23 - 9$

Solution:

14

Exercise:

Problem: $46 - 21$

Exercise:

Problem: $82 - 59$

Solution:

23

Exercise:

Problem: $110 - 87$

Exercise:

Problem: $539 - 217$

Solution:

322

Exercise:

Problem: $415 - 296$

Exercise:

Problem: $1{,}020 - 640$

Solution:

380

Exercise:

Problem: $8{,}355 - 3{,}947$

Exercise:

Problem: $10{,}000 - 15$

Solution:

9,985

Exercise:

Problem: $54{,}925 - 35{,}647$

Translate Word Phrases to Math Notation

In the following exercises, translate and simplify.

Exercise:

Problem: the difference of nineteen and thirteen

Solution:

$19 - 13; 6$

Exercise:

Problem: subtract sixty-five from one hundred

Exercise:

Problem: seventy-four decreased by eight

Solution:

$74 - 8$; 66

Exercise:

Problem: twenty-three less than forty-one

Subtract Whole Numbers in Applications

In the following exercises, solve.

Exercise:

Problem:

Temperature The high temperature in Peoria one day was 86 degrees Fahrenheit and the low temperature was 28 degrees Fahrenheit. What was the difference between the high and low temperatures?

Solution:

58 degrees Fahrenheit

Exercise:

Problem:

Savings Lynn wants to go on a cruise that costs $\$2,485.$ She has $\$948$ in her vacation savings account. How much more does she need to save in order to pay for the cruise?

Multiply Whole Numbers

Use Multiplication Notation

In the following exercises, translate from math notation to words.

Exercise:

Problem: 8×5

Solution:

eight times five the product of eight and five

Exercise:

Problem: $6 \cdot 14$

Exercise:

Problem: $(10)(95)$

Solution:

ten times ninety-five; the product of ten and ninety-five

Exercise:

Problem: $54(72)$

Model Multiplication of Whole Numbers

In the following exercises, model the multiplication.

Exercise:

Problem: 2×4

Solution:

Exercise:

Problem: 3×8

Multiply Whole Numbers

In the following exercises, fill in the missing values in each chart.

Exercise:

Problem:

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0		0		0	0
1	0	1	2		4	5	6	7		9
2	0		4		8	10		14	16	
3		3		9			18		24	
4	0	4		12			24			36
5	0	5	10		20		30	35	40	45
6			12	18			36	42		54
7	0	7		21		35			56	63
8	0	8	16		32		48		64	
9			18	27	36			63	72	

Solution:

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Exercise:

Problem:

×	3	4	5	6	7	8	9
6							
7							
8							
9							

In the following exercises, multiply.

Exercise:

Problem: $0 \cdot 14$

Solution:

0

Exercise:

Problem: $(256)0$

Exercise:

Problem: $1 \cdot 99$

Solution:

99

Exercise:

Problem: $(4{,}789)1$

Exercise:

Problem: ⓐ $7 \cdot 4$ ⓑ $4 \cdot 7$

Solution:

ⓐ 28
ⓑ 28

Exercise:

Problem: $(25)(6)$

Exercise:

Problem: $9{,}261 \times 3$

Solution:

27,783

Exercise:

Problem: $48 \cdot 76$

Exercise:

Problem: $64 \cdot 10$

Solution:

640

Exercise:

Problem: $1{,}000(22)$

Exercise:

Problem: 162×493

Solution:

79,866

Exercise:

Problem: $(601)(943)$

Exercise:

Problem: $3{,}624 \times 517$

Solution:

1,873,608

Exercise:

Problem: $10{,}538 \cdot 22$

Translate Word Phrases to Math Notation

In the following exercises, translate and simplify.

Exercise:

Problem: the product of 15 and 28

Solution:

15(28); 420

Exercise:

Problem: ninety-four times thirty-three

Exercise:

Problem: twice 575

Solution:

2(575); 1,150

Exercise:

Problem: ten times two hundred sixty-four

Multiply Whole Numbers in Applications

In the following exercises, solve.

Exercise:

Problem:

Gardening Geniece bought 8 packs of marigolds to plant in her yard. Each pack has 6 flowers. How many marigolds did Geniece buy?

Solution:

48 marigolds

Exercise:

Problem:

Cooking Ratika is making rice for a dinner party. The number of cups of water is twice the number of cups of rice. If Ratika plans to use 4 cups of rice, how many cups of water does she need?

Exercise:

Problem:

Multiplex There are twelve theaters at the multiplex and each theater has 150 seats. What is the total number of seats at the multiplex?

Solution:

1,800 seats

Exercise:

Problem:

Roofing Lewis needs to put new shingles on his roof. The roof is a rectangle, 30 feet by 24 feet. What is the area of the roof?

Divide Whole Numbers

Use Division Notation

Translate from math notation to words.

Exercise:

Problem: $54 \div 9$

Solution:

fifty-four divided by nine; the quotient of fifty-four and nine

Exercise:

Problem: $42/7$

Exercise:

Problem: $\frac{72}{8}$

Solution:

seventy-two divided by eight; the quotient of seventy-two and eight

Exercise:

Problem: $6\overline{)48}$

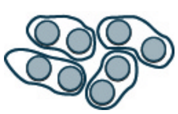
Model Division of Whole Numbers

In the following exercises, model.

Exercise:

Problem: $8 \div 2$

Solution:



Exercise:

Problem: $3\overline{)12}$

Divide Whole Numbers

In the following exercises, divide. Then check by multiplying.

Exercise:

Problem: $14 \div 2$

Solution:

7

Exercise:

Problem: $\frac{32}{8}$

Exercise:

Problem: $52 \div 4$

Solution:

13

Exercise:

Problem: $26\overline{)26}$

Exercise:

Problem: $\frac{97}{1}$

Solution:

97

Exercise:

Problem: $0 \div 52$

Exercise:

Problem: $100 \div 0$

Solution:

undefined

Exercise:

Problem: $\frac{355}{5}$

Exercise:

Problem: $3828 \div 6$

Solution:

638

Exercise:

Problem: $31\overline{)1{,}519}$

Exercise:

Problem: $\frac{7505}{25}$

Solution:

300 R5

Exercise:

Problem: $5{,}166 \div 42$

Translate Word Phrases to Math Notation

In the following exercises, translate and simplify.

Exercise:

Problem: the quotient of 64 and 16

Solution:

$64 \div 16$; 4

Exercise:

Problem: the quotient of 572 and 52

Divide Whole Numbers in Applications

In the following exercises, solve.

Exercise:

Problem:

Ribbon One spool of ribbon is 27 feet. Lizbeth uses 3 feet of ribbon for each gift basket that she wraps. How many gift baskets can Lizbeth wrap from one spool of ribbon?

Solution:

9 baskets

Exercise:

Problem:

Juice One carton of fruit juice is 128 ounces. How many 4 ounce cups can Shayla fill from one carton of juice?

Chapter Practice Test

Exercise:

Problem: Determine which of the following numbers are

ⓐ counting numbers
ⓑ whole numbers.

$0, 4, 87$

Solution:

ⓐ 4, 87
ⓑ 0, 4, 8

Exercise:

Problem: Find the place value of the given digits in the number $549{,}362$.

ⓐ 9
ⓑ 6
ⓒ 2
ⓓ 5

Exercise:

Problem: Write each number as a whole number using digits.

ⓐ six hundred thirteen
ⓑ fifty-five thousand two hundred eight

Solution:

ⓐ 613
ⓑ 55,208

Exercise:

Problem: Round $25{,}849$ to the nearest hundred.

Simplify.

Exercise:

Problem: $45 + 23$

Solution:

68

Exercise:

Problem: $65 - 42$

Exercise:

Problem: $85 \div 5$

Solution:

17

Exercise:

Problem: $1{,}000 \times 8$

Exercise:

Problem: $90 - 58$

Solution:

32

Exercise:

Problem: $73 + 89$

Exercise:

Problem: $(0)(12{,}675)$

Solution:

0

Exercise:

Problem: $634 + 255$

Exercise:

Problem: $\frac{0}{9}$

Solution:

0

Exercise:

Problem: $8\overline{)128}$

Exercise:

Problem: $145 - 79$

Solution:

66

Exercise:

Problem: $299 + 836$

Exercise:

Problem: $7 \cdot 475$

Solution:

3,325

Exercise:

Problem: $8{,}528 + 704$

Exercise:

Problem: $35(14)$

Solution:

490

Exercise:

Problem: $\frac{26}{0}$

Exercise:

Problem: $733 - 291$

Solution:

442

Exercise:

Problem: $4{,}916 - 1{,}538$

Exercise:

Problem: $495 \div 45$

Solution:

11

Exercise:

Problem: 52×983

Translate each phrase to math notation and then simplify.

Exercise:

Problem: The sum of 16 and 58

Solution:

16 + 58; 74

Exercise:

Problem: The product of 9 and 15

Exercise:

Problem: The difference of 32 and 18

Solution:

32 − 18; 14

Exercise:

Problem: The quotient of 63 and 21

Exercise:

Problem: Twice 524

Solution:

2(524); 1,048

Exercise:

Problem: 29 more than 32

Exercise:

Problem: 50 less than 300

Solution:

300 − 50; 250

In the following exercises, solve.

Exercise:

Problem:

LaVelle buys a jumbo bag of 84 candies to make favor bags for her son's party. If she wants to make 12 bags, how many candies should she put in each bag?

Exercise:

Problem:

Last month, Stan's take-home pay was $\$3,816$ and his expenses were $\$3,472$. How much of his take-home pay did Stan have left after he paid his expenses?

Solution:

Stan had $344 left.

Exercise:

Problem:

Each class at Greenville School has 22 children enrolled. The school has 24 classes. How many children are enrolled at Greenville School?

Exercise:

Problem:

Clayton walked 12 blocks to his mother's house, 6 blocks to the gym, and 9 blocks to the grocery store before walking the last 3 blocks home. What was the total number of blocks that Clayton walked?

Solution:

Clayton walked 30 blocks.

Glossary

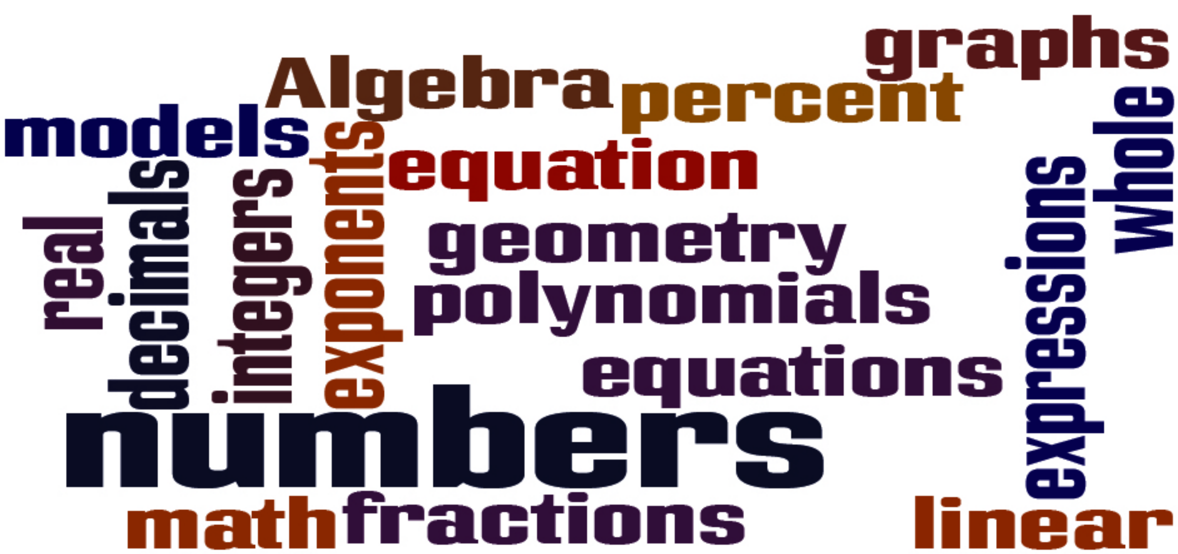
dividend
: When dividing two numbers, the dividend is the number being divided.

divisor
: When dividing two numbers, the divisor is the number dividing the dividend.

quotient
: The quotient is the result of dividing two numbers.

Introduction to the Language of Algebra
class="introduction"
--

Algebra has a language of its own. The picture shows just some of the words you may see and use in your study of Prealgebra.



You may not realize it, but you already use algebra every day. Perhaps you figure out how much to tip a server in a restaurant. Maybe you calculate the amount of change you should get when you pay for something. It could

even be when you compare batting averages of your favorite players. You can describe the algebra you use in specific words, and follow an orderly process. In this chapter, you will explore the words used to describe algebra and start on your path to solving algebraic problems easily, both in class and in your everyday life.

Use the Language of Algebra
By the end of this section, you will be able to:

- Use variables and algebraic symbols
- Identify expressions and equations
- Simplify expressions with exponents
- Simplify expressions using the order of operations

Note:
Before you get started, take this readiness quiz.

1. Add: $43+69$.
   If you missed this problem, review [link].
2. Multiply: $(896)201$.
   If you missed this problem, review [link].
3. Divide: $7{,}263\div 9$.
   If you missed this problem, review [link].

Use Variables and Algebraic Symbols

Greg and Alex have the same birthday, but they were born in different years. This year Greg is 20 years old and Alex is 23, so Alex is 3 years older than Greg. When Greg was 12, Alex was 15. When Greg is 35, Alex will be 38. No matter what Greg's age is, Alex's age will always be 3 years more, right?

In the language of algebra, we say that Greg's age and Alex's age are variable and the three is a constant. The ages change, or vary, so age is a variable. The 3 years between them always stays the same, so the age difference is the constant.

Don't confuse the word "vary" with the word "very". The words sound alike but are spelled differently. **"Vary"** means that something can change or have different values. **"Very"** means extremely or highly. Try to explain the difference between these two small paragraphs.

1. How did you do on the test? I did **very** well.
   The person got a high score on the test.
2. How did you do on the test? It **varies**. I did well on the first and last parts but had trouble with the middle portion.
   The person got a high score on the first and last portions of the test but had a low or average score with the middle portion.

In algebra, letters of the alphabet are used to represent variables. Suppose we call Greg’s age g. Then we could use $g+3$ to represent Alex’s age. See [link].

Greg’s age	Alex’s age
12	15
20	23
35	38
g	$g+3$

Letters are used to represent variables. Letters often used for variables are x, y, a, b, and c.

Note:
Variables and Constants
A variable is a letter that represents a number or quantity whose value may change.
A constant is a number whose value always stays the same.

To write algebraically, we need some symbols as well as numbers and variables. There are several types of symbols we will be using. In Whole Numbers, we introduced the symbols for the four basic arithmetic operations: addition, subtraction, multiplication, and division. We will summarize them here, along with words we use for the operations and the result.

Operation	Notation	Say:	The result is…
Addition	$a+b$	a plus b	the sum of a and b
Subtraction	$a-b$	a minus b	the difference of a and b

Operation	**Notation**	**Say:**	**The result is…**
Multiplication	$a \cdot b, (a)(b), (a)b, a(b), \text{ab}$	a times b	The product of a and b
Division	$a \div b, a/b, \frac{a}{b}, b\overline{)a}$	a divided by b	The quotient of a and b

In algebra, the cross symbol, $\times$, is not used to show multiplication because that symbol may cause confusion. Does $3xy$ mean $3 \times y$ (three times y) or $3 \cdot x \cdot y$ (three times x times y)? To make it clear, use • or parentheses for multiplication.

Putting the variables a and b right next to each other also means multiplication. This needs to be clear in the context that it is used so that it is not misunderstood to be a new variable with the name ab.

It is very common to write a number directly to the left of a variable. This also means multiplication. For example, $3y$ means $3 \cdot y$.

We perform these operations on two numbers. When translating from symbolic form to words, or from words to symbolic form, pay attention to the words *of* or *and* to help you find the numbers.

- The *sum* ***of*** 5 ***and*** 3 means add 5 plus 3, which we write as $5+3$.
- The *difference* ***of*** 9 ***and*** 2 means subtract 9 minus 2, which we write as $9-2$.
- The *product* ***of*** 4 ***and*** 8 means multiply 4 times 8, which we can write as $4 \cdot 8$.
- The *quotient* ***of*** 20 ***and*** 5 means divide 20 by 5, which we can write as $20 \div 5$.

Example:
Exercise:

Problem: Translate from algebra to words:

ⓐ $12+14$
ⓑ $(30)(5)$
ⓒ $64 \div 8$
ⓓ $x-y$

Solution:
Solution

ⓐ

$12 + 14$

12 plus 14

the sum of twelve and fourteen

ⓑ

$(30)(5)$

30 times 5

the product of thirty and five

ⓒ

$64 \div 8$

64 divided by 8

the quotient of sixty-four and eight

ⓓ

$x - y$

x minus y
the difference of x and y

Note:
Exercise:

Problem: Translate from algebra to words.

ⓐ $18 + 11$
ⓑ $(27)(9)$
ⓒ $84 \div 7$
ⓓ $p - q$

Solution:

ⓐ 18 plus 11; the sum of eighteen and eleven
ⓑ 27 times 9; the product of twenty-seven and nine
ⓒ 84 divided by 7; the quotient of eighty-four and seven
ⓓ *p* minus *q*; the difference of *p* and *q*

Note:
Exercise:

Problem: Translate from algebra to words.

ⓐ $47 - 19$
ⓑ $72 \div 9$
ⓒ $m + n$
ⓓ $(13)(7)$

Solution:

ⓐ 47 minus 19; the difference of forty-seven and nineteen
ⓑ 72 divided by 9; the quotient of seventy-two and nine
ⓒ *m* plus *n*; the sum of *m* and *n*

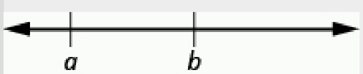
ⓓ 13 times 7; the product of thirteen and seven

When two quantities have the same value, we say they are equal and connect them with an *equal sign*.

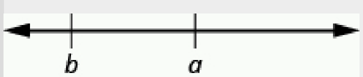
Note:
Equality Symbol
$a=b$ is read a is equal to b
The symbol $=$ is called the equal sign.

An inequality is used in algebra to compare two quantities that may have different values. The number line can help you understand inequalities. Remember that on the number line the numbers get larger as they go from left to right. So if we know that b is greater than a, it means that b is to the right of a on the number line. We use the symbols "$<$" and "$>$" for inequalities.

Note:
Inequality
$a<b$ is read a is less than b
a is to the left of b on the number line



$a>b$ is read a is greater than b
a is to the right of b on the number line



The expressions $a<b$ and $a>b$ can be read from left-to-right or right-to-left, though in English we usually read from left-to-right. In general,
Equation:

$a < b$ is equivalent to $b > a$. For example, $7 < 11$ is equivalent to $11 > 7$.
$a > b$ is equivalent to $b < a$. For example, $17 > 4$ is equivalent to $4 < 17$.

When we write an inequality symbol with a line under it, such as $a \le b$, it means $a < b$ or $a = b$. We read this a is less than or equal to b. Also, if we put a slash through an equal sign, $\ne$, it means not equal.

We summarize the symbols of equality and inequality in [link].

Algebraic Notation	**Say**
$a = b$	a is equal to b
$a \ne b$	a is not equal to b
$a < b$	a is less than b
$a > b$	a is greater than b
$a \le b$	a is less than or equal to b
$a \ge b$	a is greater than or equal to b

Note:
Symbols $<$ and $>$
The symbols $<$ and $>$ each have a smaller side and a larger side.
smaller side $<$ larger side
larger side $>$ smaller side
The smaller side of the symbol faces the smaller number and the larger faces the larger number.

Example:
Exercise:

Problem: Translate from algebra to words:

ⓐ $20 \leq 35$
ⓑ $11 \neq 15 - 3$
ⓒ $9 > 10 \div 2$
ⓓ $x + 2 < 10$

Solution:
Solution

ⓐ
$20 \leq 35$
20 is less than or equal to 35

ⓑ
$11 \neq 15 - 3$
11 is not equal to 15 minus 3

ⓒ
$9 > 10 \div 2$
9 is greater than 10 divided by 2

ⓓ
$x + 2 < 10$
x plus 2 is less than 10

Note:
Exercise:

Problem: Translate from algebra to words.

ⓐ $14 \leq 27$
ⓑ $19 - 2 \neq 8$
ⓒ $12 > 4 \div 2$
ⓓ $x - 7 < 1$

Solution:

ⓐ fourteen is less than or equal to twenty-seven
ⓑ nineteen minus two is not equal to eight
ⓒ twelve is greater than four divided by two
ⓓ *x* minus seven is less than one

Note:
Exercise:

Problem: Translate from algebra to words.

ⓐ $19 \geq 15$
ⓑ $7 = 12 - 5$
ⓒ $15 \div 3 < 8$
ⓓ $y - 3 > 6$

Solution:

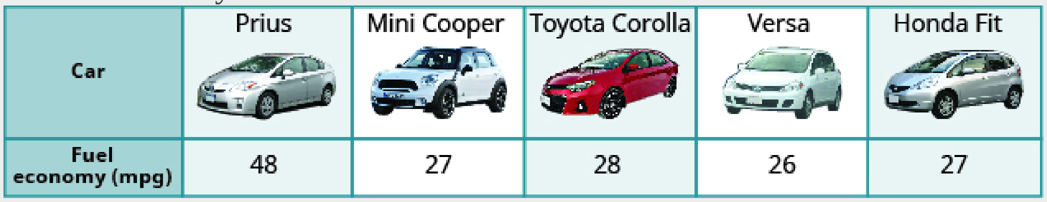
ⓐ nineteen is greater than or equal to fifteen

ⓑ seven is equal to twelve minus five
ⓒ fifteen divided by three is less than eight
ⓓ y minus three is greater than six

Example:
Exercise:

Problem:

The information in [link] compares the fuel economy in miles-per-gallon (mpg) of several cars. Write the appropriate symbol $=$, $<$, or $>$ in each expression to compare the fuel economy of the cars.



Car	Prius	Mini Cooper	Toyota Corolla	Versa	Honda Fit
Fuel economy (mpg)	48	27	28	26	27

(credit: modification of work by Bernard Goldbach, Wikimedia Commons)

ⓐ MPG of Prius_____ MPG of Mini Cooper
ⓑ MPG of Versa_____ MPG of Fit
ⓒ MPG of Mini Cooper_____ MPG of Fit
ⓓ MPG of Corolla_____ MPG of Versa
ⓔ MPG of Corolla_____ MPG of Prius

Solution:
Solution

ⓐ	
	MPG of Prius____MPG of Mini Cooper
Find the values in the chart.	48____27

Compare.	48 > 27
	MPG of Prius > MPG of Mini Cooper

ⓑ	
	MPG of Versa____MPG of Fit
Find the values in the chart.	26____27
Compare.	26 < 27
	MPG of Versa < MPG of Fit

ⓒ	
	MPG of Mini Cooper____MPG of Fit
Find the values in the chart.	27____27
Compare.	27 = 27
	MPG of Mini Cooper = MPG of Fit

ⓓ	
	MPG of Corolla____MPG of Versa

Find the values in the chart.	28____26
Compare.	28 > 26
	MPG of Corolla > MPG of Versa

ⓔ	
	MPG of Corolla____MPG of Prius
Find the values in the chart.	28____48
Compare.	28 < 48
	MPG of Corolla < MPG of Prius

Note:
Exercise:

Problem: Use [link] to fill in the appropriate symbol, $=$, $<$, or $>$.

ⓐ MPG of Prius_____MPG of Versa
ⓑ MPG of Mini Cooper_____ MPG of Corolla

Solution:

ⓐ >
ⓑ >

Note:
Exercise:

Problem: Use [link] to fill in the appropriate symbol, $=$, $<$, or $>$.

ⓐ MPG of Fit_____ MPG of Prius
ⓑ MPG of Corolla _____ MPG of Fit

Solution:

ⓐ <
ⓑ <

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in written language. They indicate which expressions are to be kept together and separate from other expressions. [link] lists three of the most commonly used grouping symbols in algebra.

Common Grouping Symbols	
parentheses	( )
brackets	[ ]
braces	{ }

Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.
Equation:

$$8(14-8) \qquad 21-3[2+4(9-8)] \qquad 24\div\{13-2[1(6-5)+4]\}$$

Identify Expressions and Equations

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement.

“Running very fast” is a phrase, but “The football player was running very fast” is a sentence. A sentence has a subject and a verb.

In algebra, we have *expressions* and *equations*. An expression is like a phrase. Here are some examples of expressions and how they relate to word phrases:

Expression	Words	Phrase
$3+5$	3 plus 5	the sum of three and five
$n-1$	n minus one	the difference of n and one
$6\cdot 7$	6 times 7	the product of six and seven
$\frac{x}{y}$	x divided by y	the quotient of x and y

Notice that the phrases do not form a complete sentence because the phrase does not have a verb. An equation is two expressions linked with an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb. Here are some examples of equations:

Equation	Sentence
$3+5=8$	The sum of three and five is equal to eight.
$n-1=14$	n minus one equals fourteen.
$6\cdot 7=42$	The product of six and seven is equal to forty-two.
$x=53$	x is equal to fifty-three.
$y+9=2y-3$	y plus nine is equal to two y minus three.

Note:
Expressions and Equations
An **expression** is a number, a variable, or a combination of numbers and variables and operation symbols.
An **equation** is made up of two expressions connected by an equal sign.

Example:
Exercise:

Problem: Determine if each is an expression or an equation:

ⓐ $16 - 6 = 10$
ⓑ $4 \cdot 2 + 1$
ⓒ $x \div 25$
ⓓ $y + 8 = 40$

Solution:
Solution

ⓐ $16 - 6 = 10$	This is an equation—two expressions are connected with an equal sign.
ⓑ $4 \cdot 2 + 1$	This is an expression—no equal sign.
ⓒ $x \div 25$	This is an expression—no equal sign.
ⓓ $y + 8 = 40$	This is an equation—two expressions are connected with an equal sign.

Note:
Exercise:

Problem: Determine if each is an expression or an equation:

ⓐ $23 + 6 = 29$
ⓑ $7 \cdot 3 - 7$

Solution:

ⓐ equation
ⓑ expression

Note:
Exercise:

Problem: Determine if each is an expression or an equation:

$y \div 14$
$x - 6 = 21$

Solution:

ⓐ expression
ⓑ equation

Simplify Expressions with Exponents

To simplify a numerical expression means to do all the math possible. For example, to simplify $4 \cdot 2 + 1$ we'd first multiply $4 \cdot 2$ to get 8 and then add the 1 to get 9. A good habit to develop is to work down the page, writing each step of the process below the previous step. The example just described would look like this:

Equation:

$$4 \cdot 2 + 1$$

Equation:

$$8 + 1$$

Equation:

$$9$$

Suppose we have the expression $2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2$. We could write this more compactly using exponential notation. Exponential notation is used in algebra to represent a product resulting from one or more identical factors. We write $2\cdot2\cdot2$ as 2^3 and $2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2$ as 2^9. In expressions such as 2^3, the 2 is called the **base** and the 3 is called the exponent. The exponent tells us how many factors of the base we have to multiply.

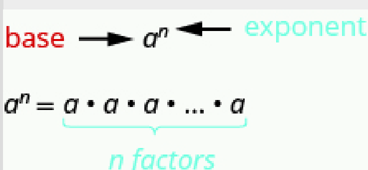
Equation:

$$\text{means multiply three factors of } 2$$

We say 2^3 is in exponential notation and $2\cdot2\cdot2$ is in expanded notation.

Note:
Exponential Notation
For any expression a^n, a is a factor multiplied by itself n times if n is a positive integer.
Equation:

$$a^n \text{means multiply } n \text{ factors of } a$$



The expression a^n is read a to the n^{th} power.

For powers of $n=2$ and $n=3$, we have special names. This is because the area of a square with a side of length a is a times a, and the volume of a cube with an edge of length a is a times a times a.
Equation:

$$a^2 \text{ is read as ''} a \text{ squared''}$$
$$a^3 \text{ is read as ''} a \text{ cubed''}$$

[link] lists some examples of expressions written in exponential notation.

Exponential Notation	In Words
7^2	7 to the second power, or 7 squared
5^3	5 to the third power, or 5 cubed
9^4	9 to the fourth power
12^5	12 to the fifth power

Example:
Exercise:

Problem: Write each expression in exponential form:

ⓐ $16 \cdot 16 \cdot 16 \cdot 16 \cdot 16 \cdot 16 \cdot 16$
ⓑ $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$
ⓒ $x \cdot x \cdot x \cdot x$
ⓓ $a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$

Solution:
Solution

ⓐ The base 16 is a factor 7 times.	16^7
ⓑ The base 9 is a factor 5 times.	9^5
ⓒ The base x is a factor 4 times.	x^4
ⓓ The base a is a factor 8 times.	a^8

Note:
Exercise:

Problem: Write each expression in exponential form:

$41 \cdot 41 \cdot 41 \cdot 41 \cdot 41$

Solution:

41^5

Note:
Exercise:

Problem: Write each expression in exponential form:

$7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$

Solution:

7^9

Example:
Exercise:

Problem: Write each exponential expression in expanded form:

ⓐ 8^6
ⓑ x^5

Solution:
Solution

ⓐ The base is 8 and the exponent is 6, so 8^6 means $8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8$

ⓑ The base is x and the exponent is 5, so x^5 means $x \cdot x \cdot x \cdot x \cdot x$

Note:
Exercise:

Problem: Write each exponential expression in expanded form:

ⓐ 4^8
ⓑ a^7

Solution:

ⓐ $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
ⓑ $a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$

Note:
Exercise:

Problem: Write each exponential expression in expanded form:

ⓐ 8^8
ⓑ b^6

Solution:

ⓐ $8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8$
ⓑ $b \cdot b \cdot b \cdot b \cdot b \cdot b$

To simplify an exponential expression without using a calculator, we write it in expanded form and then multiply the factors.

Example:
Exercise:

Problem: Simplify: 3^4.

Solution:
Solution

	3^4
Expand the expression.	$3 \cdot 3 \cdot 3 \cdot 3$
Multiply left to right.	$9 \cdot 3 \cdot 3$
	$27 \cdot 3$
Multiply.	81

Note:
Exercise:

Problem: Simplify:

ⓐ 5^3
ⓑ 1^7

Solution:

ⓐ 125
ⓑ 1

Note:
Exercise:

Problem: Simplify:

ⓐ 7^2
ⓑ 0^5

Solution:

ⓐ 49
ⓑ 0

Simplify Expressions Using the Order of Operations

We’ve introduced most of the symbols and notation used in algebra, but now we need to clarify the order of operations. Otherwise, expressions may have different meanings, and they may result in different values.

For example, consider the expression:

Equation:

$$4 + 3 \cdot 7$$

Equation:

Some students say it simplifies to 49.		Some students say it simplifies to 25.	
	$4 + 3 \cdot 7$		$4 + 3 \cdot 7$
Since $4 + 3$ gives 7.	$7 \cdot 7$	Since $3 \cdot 7$ is 21.	$4 + 21$
And $7 \cdot 7$ is 49.	49	And $21 + 4$ makes 25.	25

Imagine the confusion that could result if every problem had several different correct answers. The same expression should give the same result. So mathematicians established some guidelines called the order of operations, which outlines the order in which parts of an expression must be simplified.

Note:
Order of Operations
When simplifying mathematical expressions perform the operations in the following order:
1. **P**arentheses and other Grouping Symbols

- Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.

2. **E**xponents

- Simplify all expressions with exponents.

3. **M**ultiplication and **D**ivision

- Perform all multiplication and division in order from left to right. These operations have equal priority.

4. **A**ddition and **S**ubtraction

- Perform all addition and subtraction in order from left to right. These operations have equal priority.

Students often ask, “How will I remember the order?”

Parentheses and other grouping symbols exist just so we can easily change the order operations are performed so what is inside them must be done first.

The key to the rest of the order is to remember to do the simpler operation after the more complex operation. We defined exponents in terms of multiplication so multiplication is simpler. Division is the inverse operation for multiplication so those are equally complex. Similarly, we defined multiplication in terms of addition so addition is simpler. Subtraction is the inverse operation for addition so those are equally complex.

The result is: Parentheses before Exponents, Exponents before Multiplication and Division, and Multiplication and Division before Addition and Subtraction

A final rule is for operations of the same complexity, we do them from left to right.

If you are unsure of the order of operations for something you want to write, you can always use parenthesis to force the computation to be in the order you want.

Example:
Exercise:

Problem: Simplify the expressions:

ⓐ $4 + 3 \cdot 7$
ⓑ $(4 + 3) \cdot 7$

Solution:
Solution

ⓐ	

	$4 + 3 \cdot 7$
Are there any **p**arentheses? No.	
Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply first.	$4 + 3 \cdot 7$
Add.	$4 + 21$
	25

ⓑ	
	$(4 + 3) \cdot 7$
Are there any **p**arentheses? Yes.	$(4 + 3) \cdot 7$
Simplify inside the parentheses.	$(7)7$
Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply.	49

Note:
Exercise:

Problem: Simplify the expressions:

ⓐ $12 - 5 \cdot 2$
ⓑ $(12 - 5) \cdot 2$

Solution:

ⓐ 2
ⓑ 14

Note:
Exercise:

Problem: Simplify the expressions:

ⓐ $8 + 3 \cdot 9$
ⓑ $(8 + 3) \cdot 9$

Solution:

ⓐ 35
ⓑ 99

Example:
Exercise:

Problem: Simplify:

ⓐ $18 \div 9 \cdot 2$
ⓑ $18 \cdot 9 \div 2$

Solution:
Solution

ⓐ	
	$18 \div 9 \cdot 2$
Are there any **p**arentheses? No.	
Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply and divide from left to right. Divide.	$2 \cdot 2$
Multiply.	4

ⓑ	
	$18 \cdot 9 \div 2$
Are there any **p**arentheses? No.	
Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply and divide from left to right.	
Multiply.	$162 \div 2$
Divide.	81

Note:
Exercise:

Problem: Simplify:

$42 \div 7 \cdot 3$

Solution:

18

Note:
Exercise:

Problem: Simplify:

$12 \cdot 3 \div 4$

Solution:

9

Example:
Exercise:

Problem: Simplify: $18 \div 6 + 4(5 - 2)$.

Solution:
Solution

	$18 \div 6 + 4(5 - 2)$

Parentheses? Yes, subtract first.	$18 \div 6 + 4(3)$
Exponents? No.	
Multiplication or division? Yes.	
Divide first because we multiply and divide left to right.	$3 + 4(3)$
Any other multiplication or division? Yes.	
Multiply.	$3 + 12$
Any other multiplication or division? No.	
Any addition or subtraction? Yes.	15

Note:
Exercise:

Problem: Simplify:

$30 \div 5 + 10(3 - 2)$

Solution:

16

Note:
Exercise:

Problem: Simplify:

$70 \div 10 + 4(6 - 2)$

Solution:

23

When there are multiple grouping symbols, we simplify the innermost parentheses first and work outward.

Example:
Exercise:

Problem: Simplify: $5 + 2^3 + 3\left[6 - 3(4 - 2)\right]$.

Solution:
Solution

	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Are there any parentheses (or other grouping symbol)? Yes.	
Focus on the parentheses that are inside the brackets.	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Subtract.	$5 + 2^3 + 3[6 - 3(2)]$
Continue inside the brackets and multiply.	$5 + 2^3 + 3[6 - 6]$
Continue inside the brackets and subtract.	$5 + 2^3 + 3[0]$

The expression inside the brackets requires no further simplification.	
Are there any exponents? Yes.	
Simplify exponents.	$5 + 2^3 + 3[0]$
Is there any multiplication or division? Yes.	
Multiply.	$5 + 8 + 3[0]$
Is there any addition or subtraction? Yes.	
Add.	$5 + 8 + 0$
Add.	$13 + 0$
	13

Note:
Exercise:

Problem: Simplify:

$9 + 5^3 - [4(9 + 3)]$

Solution:

86

Note:

Exercise:

Problem: Simplify:

$7^2 - 2\left[4(5+1)\right]$

Solution:

1

Example:
Exercise:

Problem: Simplify: $2^3 + 3^4 \div 3 - 5^2$.

Solution:
Solution

	$2^3 + 3^4 \div 3 - 5^2$
If an expression has several exponents, they may be simplified in the same step.	
Simplify exponents.	$2^3 + 3^4 \div 3 - 5^2$
Divide.	$8 + 81 \div 3 - 25$
Add.	$8 + 27 - 25$
Subtract.	$35 - 25$

10

Note:
Exercise:

Problem: Simplify:

$3^2 + 2^4 \div 2 + 4^3$

Solution:

81

Note:
Exercise:

Problem: Simplify:

$6^2 - 5^3 \div 5 + 8^2$

Solution:

75

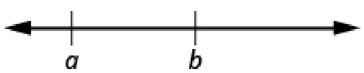
Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Order of Operations
- Order of Operations – The Basics
- Ex: Evaluate an Expression Using the Order of Operations
- Example 3: Evaluate an Expression Using The Order of Operations

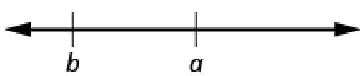
Key Concepts

Operation	**Notation**	**Say:**	**The result is…**
Addition	$a+b$	a plus b	the sum of a and b
Multiplication	$a \cdot b, (a)(b), (a)b, a(b)$	a times b	The product of a and b
Subtraction	$a-b$	a minus b	the difference of a and b
Division	$a \div b, a/b, \frac{a}{b}, b\overline{)a}$	a divided by b	The quotient of a and b

- **Equality Symbol**
    - $a=b$ is read as a is equal to b
    - The symbol $=$ is called the equal sign.
- **Inequality**
    - $a<b$ is read a is less than b
    - a is to the left of b on the number line



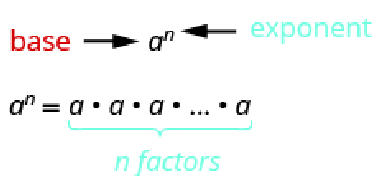
    - $a>b$ is read a is greater than b
    - a is to the right of b on the number line



Algebraic Notation	**Say**
$a=b$	a is equal to b
$a \neq b$	a is not equal to b
$a<b$	a is less than b

Algebraic Notation	Say
$a > b$	a is greater than b
$a \leq b$	a is less than or equal to b
$a \geq b$	a is greater than or equal to b

- **Exponential Notation**
  - For any expression a^n is a factor multiplied by itself n times, if n is a positive integer.
  - a^n means multiply n factors of a



  - The expression of a^n is read a to the $n\text{th}$ power.

Order of Operations When simplifying mathematical expressions perform the operations in the following order:

- Parentheses and other Grouping Symbols: Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
- Exponents: Simplify all expressions with exponents.
- Multiplication and Division: Perform all multiplication and division in order from left to right. These operations have equal priority.
- Addition and Subtraction: Perform all addition and subtraction in order from left to right. These operations have equal priority.

Practice Makes Perfect

Use Variables and Algebraic Symbols

In the following exercises, translate from algebraic notation to words.
Exercise:

Problem: $16 - 9$

Solution:

16 minus 9, the difference of sixteen and nine

Exercise:

Problem: $25 - 7$

Exercise:

Problem: $5 \cdot 6$

Solution:

5 times 6, the product of five and six

Exercise:

Problem: $3 \cdot 9$

Exercise:

Problem: $28 \div 4$

Solution:

28 divided by 4, the quotient of twenty-eight and four

Exercise:

Problem: $45 \div 5$

Exercise:

Problem: $x + 8$

Solution:

x plus 8, the sum of *x* and eight

Exercise:

Problem: $x + 11$

Exercise:

Problem: $(2)(7)$

Solution:

2 times 7, the product of two and seven

Exercise:

Problem: $(4)(8)$

Exercise:

Problem: $14 < 21$

Solution:

fourteen is less than twenty-one

Exercise:

Problem: $17 < 35$

Exercise:

Problem: $36 \geq 19$

Solution:

thirty-six is greater than or equal to nineteen

Exercise:

Problem: $42 \geq 27$

Exercise:

Problem: $3n = 24$

Solution:

3 times n equals 24, the product of three and n equals twenty-four

Exercise:

Problem: $6n = 36$

Exercise:

Problem: $y - 1 > 6$

Solution:

y minus 1 is greater than 6, the difference of *y* and one is greater than six

Exercise:

Problem: $y - 4 > 8$

Exercise:

Problem: $2 \leq 18 \div 6$

Solution:

2 is less than or equal to 18 divided by 6; 2 is less than or equal to the quotient of eighteen and six

Exercise:

Problem: $3 \leq 20 \div 4$

Exercise:

Problem: $a \neq 7 \cdot 4$

Solution:

a is not equal to 7 times 4, *a* is not equal to the product of seven and four

Exercise:

Problem: $a \neq 1 \cdot 12$

Identify Expressions and Equations

In the following exercises, determine if each is an expression or an equation.

Exercise:

Problem: $9 \cdot 6 = 54$

Solution:

equation

Exercise:

Problem: $7 \cdot 9 = 63$

Exercise:

Problem: $5 \cdot 4 + 3$

Solution:

expression

Exercise:

Problem: $6 \cdot 3 + 5$

Exercise:

Problem: $x + 7$

Solution:

expression

Exercise:

Problem: $x + 9$

Exercise:

Problem: $y - 5 = 25$

Solution:

equation

Exercise:

Problem: $y - 8 = 32$

Simplify Expressions with Exponents

In the following exercises, write in exponential form.

Exercise:

Problem: $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

Solution:

3^7

Exercise:

Problem: $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

Exercise:

Problem: $x \cdot x \cdot x \cdot x \cdot x$

Solution:

x^5

Exercise:

Problem: $y \cdot y \cdot y \cdot y \cdot y \cdot y$

In the following exercises, write in expanded form.

Exercise:

Problem: 5^3

Solution:

125

Exercise:

Problem: 8^3

Exercise:

Problem: 2^8

Solution:

256

Exercise:

Problem: 10^5

Simplify Expressions Using the Order of Operations

In the following exercises, simplify.

Exercise:

Problem:

ⓐ $3+8\cdot 5$
ⓑ $(3{+}8)\cdot 5$

Solution:

ⓐ 43
ⓑ 55

Exercise:

Problem:

ⓐ $2+6\cdot 3$
ⓑ $(2{+}6)\cdot 3$

Exercise:

Problem: $2^3-12\div(9-5)$

Solution:

5

Exercise:

Problem: $3^2-18\div(11-5)$

Exercise:

Problem: $3\cdot 8+5\cdot 2$

Solution:

34

Exercise:

Problem: $4\cdot 7+3\cdot 5$

Exercise:

Problem: $2+8(6+1)$

Solution:

58

Exercise:

Problem: $4 + 6(3 + 6)$

Exercise:

Problem: $4 \cdot 12/8$

Solution:

6

Exercise:

Problem: $2 \cdot 36/6$

Exercise:

Problem: $6 + 10/2 + 2$

Solution:

13

Exercise:

Problem: $9 + 12/3 + 4$

Exercise:

Problem: $(6 + 10) \div (2 + 2)$

Solution:

4

Exercise:

Problem: $(9 + 12) \div (3 + 4)$

Exercise:

Problem: $20 \div 4 + 6 \cdot 5$

Solution:

35

Exercise:

Problem: $33 \div 3 + 8 \cdot 2$

Exercise:

Problem: $20 \div (4 + 6) \cdot 5$

Solution:

10

Exercise:

Problem: $33 \div (3 + 8) \cdot 2$

Exercise:

Problem: $4^2 + 5^2$

Solution:

41

Exercise:

Problem: $3^2 + 7^2$

Exercise:

Problem: $(4 + 5)^2$

Solution:

81

Exercise:

Problem: $(3 + 7)^2$

Exercise:

Problem: $3(1 + 9 \cdot 6) - 4^2$

Solution:

149

Exercise:

Problem: $5(2+8\cdot 4)-7^2$

Exercise:

Problem: $2[1+3(10-2)]$

Solution:

50

Exercise:

Problem: $5[2+4(3-2)]$

Everyday Math

Exercise:

Problem:

Basketball In the 2014 NBA playoffs, the San Antonio Spurs beat the Miami Heat. The table below shows the heights of the starters on each team. Use this table to fill in the appropriate symbol $(=,<,>)$.

Spurs	Height		Heat	Height
Tim Duncan	$83''$		Rashard Lewis	$82''$
Boris Diaw	$80''$		LeBron James	$80''$
Kawhi Leonard	$79''$		Chris Bosh	$83''$
Tony Parker	$74''$		Dwyane Wade	$76''$
Danny Green	$78''$		Ray Allen	$77''$

ⓐ Height of Tim Duncan____Height of Rashard Lewis
ⓑ Height of Boris Diaw____Height of LeBron James
ⓒ Height of Kawhi Leonard____Height of Chris Bosh
ⓓ Height of Tony Parker____Height of Dwyane Wade
ⓔ Height of Danny Green____Height of Ray Allen

Exercise:

Problem:

Elevation In Colorado there are more than 50 mountains with an elevation of over $14,000\text{ feet}$. The table shows the ten tallest. Use this table to fill in the appropriate inequality symbol.

Mountain	**Elevation**
Mt. Elbert	$14,433'$
Mt. Massive	$14,421'$
Mt. Harvard	$14,420'$
Blanca Peak	$14,345'$
La Plata Peak	$14,336'$
Uncompahgre Peak	$14,309'$
Crestone Peak	$14,294'$
Mt. Lincoln	$14,286'$
Grays Peak	$14,270'$
Mt. Antero	$14,269'$

ⓐ Elevation of La Plata Peak____Elevation of Mt. Antero
ⓑ Elevation of Blanca Peak____Elevation of Mt. Elbert
ⓒ Elevation of Gray’s Peak____Elevation of Mt. Lincoln
ⓓ Elevation of Mt. Massive____Elevation of Crestone Peak

ⓔ Elevation of Mt. Harvard____Elevation of Uncompahgre Peak

Writing Exercises

Exercise:

Problem: Explain the difference between an expression and an equation.

Exercise:

Problem: Why is it important to use the order of operations to simplify an expression?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
use variables and algebraic symbols.			
identify expressions and equations.			
simplify expressions with exponents.			
simplify expressions using the order of operations.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to

discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

expressions
: An expression is a number, a variable, or a combination of numbers and variables and operation symbols.

equation
: An equation is made up of two expressions connected by an equal sign.

Evaluate, Simplify, and Translate Expressions
By the end of this section, you will be able to:

- Evaluate algebraic expressions
- Identify terms, coefficients, and like terms
- Simplify expressions by combining like terms
- Translate word phrases to algebraic expressions

Note:
Before you get started, take this readiness quiz.

1. Is $n \div 5$ an expression or an equation?
   If you missed this problem, review [link].
2. Simplify 4^5.
   If you missed this problem, review [link].
3. Simplify $1 + 8 \cdot 9$.
   If you missed this problem, review [link].

Evaluate Algebraic Expressions

In the last section, we simplified expressions using the order of operations. In this section, we'll evaluate expressions—again following the order of operations.

To **evaluate** an algebraic expression means to find the value of the expression when the variable is replaced by a given number. To evaluate an expression, we substitute the given number for the variable in the expression and then simplify the expression using the order of operations.

Example:
Exercise:

Problem: Evaluate $x + 7$ when

ⓐ $x = 3$
ⓑ $x = 12$

Solution:
Solution

ⓐ To evaluate, substitute 3 for x in the expression, and then simplify.

	$x + 7$
Substitute.	$3 + 7$
Add.	10

When $x = 3$, the expression $x + 7$ has a value of 10.

ⓑ To evaluate, substitute 12 for x in the expression, and then simplify.

	$x + 7$
Substitute.	$12 + 7$
Add.	19

When $x = 12$, the expression $x + 7$ has a value of 19.

Notice that we got different results for parts ⓐ and ⓑ even though we started with the same expression. This is because the values used for x were different. When we evaluate an expression, the value varies depending on the value used for the variable.

Note:
Exercise:

Problem: Evaluate:

$y + 4$ when

ⓐ $y = 6$
ⓑ $y = 15$

Solution:

ⓐ 10
ⓑ 19

Note:
Exercise:

Problem: Evaluate:

$a - 5$ when

ⓐ $a = 9$
ⓑ $a = 17$

Solution:

ⓐ 4
ⓑ 12

Example:
Exercise:

Problem: Evaluate $9x - 2$, when

ⓐ $x = 5$
ⓑ $x = 1$

Solution:
Solution

Remember ab means a times b, so $9x$ means 9 times x.

ⓐ To evaluate the expression when $x = 5$, we substitute 5 for x, and then simplify.

	$9x - 2$
Substitute 5 for x.	$9 \cdot 5 - 2$
Multiply.	$45 - 2$
Subtract.	43

ⓑ To evaluate the expression when $x = 1$, we substitute 1 for x, and then simplify.

	$9x - 2$
Substitute 1 for x.	$9(1) - 2$
Multiply.	$9 - 2$
Subtract.	7

Notice that in part ⓐ that we wrote $9 \cdot 5$ and in part ⓑ we wrote $9(1)$. Both the dot and the parentheses tell us to multiply.

Note:
Exercise:

Problem: Evaluate:

$8x - 3$, when

ⓐ $x = 2$
ⓑ $x = 1$

Solution:

ⓐ 13
ⓑ 5

Note:
Exercise:

Problem: Evaluate:

$4y - 4$, when

ⓐ $y = 3$
ⓑ $y = 5$

Solution:

ⓐ 8
ⓑ 16

Example:
Exercise:

Problem: Evaluate x^2 when $x = 10$.

Solution:
Solution

We substitute 10 for x, and then simplify the expression.

	x^2
Substitute 10 for x.	10^2
Use the definition of exponent.	$10 \cdot 10$
Multiply.	100

When $x = 10$, the expression x^2 has a value of 100.

Note:
Exercise:

Problem: Evaluate:

x^2 when $x = 8$.

Solution:

64

Note:
Exercise:

Problem: Evaluate:

x^3 when $x = 6$.

Solution:

216

Example:
Exercise:

Problem: Evaluate 2^x when $x = 5$.

Solution:
Solution

In this expression, the variable is an exponent.

	2^x
Substitute 5 for x.	2^5
Use the definition of exponent.	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
Multiply.	32

When $x = 5$, the expression 2^x has a value of 32.

Note:
Exercise:

Problem: Evaluate:

2^x when $x = 6$.

Solution:

64

Note:
Exercise:

Problem: Evaluate:

3^x when $x = 4$.

Solution:

81

Example:
Exercise:

Problem: Evaluate $3x + 4y - 6$ when $x = 10$ and $y = 2$.

Solution:
Solution

This expression contains two variables, so we must make two substitutions.

	$3x + 4y - 6$
Substitute 10 for x and 2 for y.	$3(10) + 4(2) - 6$
Multiply.	$30 + 8 - 6$
Add and subtract left to right.	32

When $x = 10$ and $y = 2$, the expression $3x + 4y - 6$ has a value of 32.

Note:
Exercise:

Problem: Evaluate:

$2x + 5y - 4$ when $x = 11$ and $y = 3$

Solution:

33

Note:
Exercise:

Problem: Evaluate:

$5x - 2y - 9$ when $x = 7$ and $y = 8$

Solution:

10

Example:
Exercise:

Problem: Evaluate $2x^2 + 3x + 8$ when $x = 4$.

Solution:
Solution

We need to be careful when an expression has a variable with an exponent. In this expression, $2x^2$ means $2 \cdot x \cdot x$ and is different from the expression $(2x)^2$, which means $2x \cdot 2x$.

	$2x^2 + 3x + 8$
Substitute 4 for each *x*.	$2(4)^2 + 3(4) + 8$
Simplify 4^2.	$2(16) + 3(4) + 8$
Multiply.	$32 + 12 + 8$
Add.	52

Note:
Exercise:

Problem: Evaluate:

$3x^2 + 4x + 1$ when $x = 3$.

Solution:

40

Note:
Exercise:

Problem: Evaluate:

$6x^2 - 4x - 7$ when $x = 2$.

Solution:

9

Identify Terms, Coefficients, and Like Terms

Algebraic expressions are made up of *terms*. A **term** is a constant or the product of a constant and one or more variables. Some examples of terms are $7, y, 5x^2, 9a,$ and $13xy$.

The constant that multiplies the variable(s) in a term is called the **coefficient**. We can think of the coefficient as the number *in front of* the variable. The coefficient of the term $3x$ is 3. When we write x, the

coefficient is 1, since $x = 1 \cdot x$. [link] gives the coefficients for each of the terms in the left column.

Term	Coefficient
7	7
$9a$	9
y	1
$5x^2$	5

An algebraic expression may consist of one or more terms added or subtracted. In this chapter, we will only work with terms that are added together. [link] gives some examples of algebraic expressions with various numbers of terms. Notice that we include the operation before a term with it.

Expression	Terms
7	7
y	y
$x + 7$	$x, 7$

Expression	Terms
$2x + 7y + 4$	$2x, 7y, 4$
$3x^2 + 4x^2 + 5y + 3$	$3x^2, 4x^2, 5y, 3$

Example:
Exercise:

Problem:

Identify each term in the expression $9b + 15x^2 + a + 6$. Then identify the coefficient of each term.

Solution:
Solution

The expression has four terms. They are $9b$, $15x^2$, a, and 6.

The coefficient of $9b$ is 9.

The coefficient of $15x^2$ is 15.

Remember that if no number is written before a variable, the coefficient is 1. So the coefficient of a is 1.

The coefficient of a constant is the constant, so the coefficient of 6 is 6.

Note:
Exercise:

Problem:

Identify all terms in the given expression, and their coefficients:

$4x + 3b + 2$

Solution:

The terms are $4x$, $3b$, and 2. The coefficients are 4, 3, and 2.

Note:
Exercise:

Problem:

Identify all terms in the given expression, and their coefficients:

$9a + 13a^2 + a^3$

Solution:

The terms are $9a$, $13a^2$, and a^3, The coefficients are 9, 13, and 1.

Some terms share common traits. Look at the following terms. Which ones seem to have traits in common?
Equation:

$$5x, 7, n^2, 4, 3x, 9n^2$$

Which of these terms are like terms?

- The terms 7 and 4 are both constant terms.
- The terms $5x$ and $3x$ are both terms with x.

- The terms n^2 and $9n^2$ both have n^2.

Terms are called **like terms** if they have the same variables and exponents. All constant terms are also like terms. So among the terms $5x, 7, n^2, 4, 3x, 9n^2,$

Equation:

$$7 \text{ and } 4 \text{ are like terms.}$$

Equation:

$$5x \text{ and } 3x \text{ are like terms.}$$

Equation:

$$n^2 \text{ and } 9n^2 \text{ are like terms.}$$

Note:
Like Terms
Terms that are either constants or have the same variables with the same exponents are like terms.

Example:
Exercise:

Problem: Identify the like terms:

ⓐ $y^3, 7x^2, 14, 23, 4y^3, 9x, 5x^2$
ⓑ $4x^2 + 2x + 5x^2 + 6x + 40x + 8xy$

Solution:
Solution

ⓐ $y^3, 7x^2, 14, 23, 4y^3, 9x, 5x^2$

Look at the variables and exponents. The expression contains $y^3, x^2, x,$ and constants.

The terms y^3 and $4y^3$ are like terms because they both have y^3.

The terms $7x^2$ and $5x^2$ are like terms because they both have x^2.

The terms 14 and 23 are like terms because they are both constants.

The term $9x$ does not have any like terms in this list since no other terms have the variable x raised to the power of 1.

ⓑ $4x^2 + 2x + 5x^2 + 6x + 40x + 8xy$

Look at the variables and exponents. The expression contains the terms $4x^2, 2x, 5x^2, 6x, 40x,$ and $8xy$

The terms $4x^2$ and $5x^2$ are like terms because they both have x^2.

The terms $2x, 6x,$ and $40x$ are like terms because they all have x.

The term $8xy$ has no like terms in the given expression because no other terms contain the two variables xy.

Note:
Exercise:

Problem: Identify the like terms in the list or the expression:

$9, 2x^3, y^2, 8x^3, 15, 9y, 11y^2$

Solution:

9, 15; $2x^3$ and $8x^3$, y^2, and $11y^2$

Note:
Exercise:

Problem: Identify the like terms in the list or the expression:

$4x^3 + 8x^2 + 19 + 3x^2 + 24 + 6x^3$

Solution:

$4x^3$ and $6x^3$; $8x^2$ and $3x^2$; 19 and 24

Simplify Expressions by Combining Like Terms

We can simplify an expression by combining the like terms. What do you think $3x + 6x$ would simplify to? If you thought $9x$, you would be right!

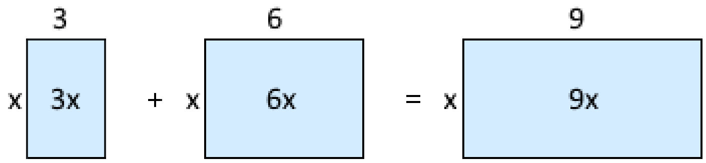
We can see why this works by writing both terms as addition problems.

$$\overbrace{x+x+x}^{3x} + \overbrace{x+x+x+x+x+x}^{6x}$$

$$9x$$

Add the coefficients and keep the same variable. It doesn't matter what x is. If you have 3 of something and add 6 more of the same thing, the result is 9 of them. For example, 3 oranges plus 6 oranges is 9 oranges.

We can also think of this in terms of three rectangles. The two smaller rectangles are 3 by x and 6 by x.
When put together they form the third rectangle, 9 by x.



This uses the distributive property of multiplication over addition: a(b + c) = ab + ac.

An equation means the expressions on the left and right side of the equal sign have the same value. It doesn't matter which side is the left and which is the right. Most people normally think of the right hand side replacing the left hand side. But it could just as easily be the other way around. Or we could rewrite the equation: ab + ac = a(b + c)

In this example, the part in common to all of the terms is x. Therefore a is x, b is 3, and c is 6.
Substituting, this fits the pattern of the distributive property, x(3) + x(6) = x(3 + 6), and x(3 + 6) = x(9).
Finally, using the commutative property of multiplications gives: 3x + 6x = 9x.

The expression $3x + 6x$ has only two terms. When an expression contains more terms, it may be helpful to rearrange the terms so that like terms are together. The Commutative Property of Addition says that we can change the order of addends without changing the sum. So we could rearrange the following expression before combining like terms.

$$3x + 4y + 2x + 6y$$

$$\underbrace{3x + 2x} + \underbrace{4y + 6y}$$

Now it is easier to see the like terms to be combined.

Note:
Combine like terms.

Identify like terms.
Rearrange the expression so like terms are together.
Add the coefficients of the like terms.

Example:
Exercise:

Problem: Simplify the expression: $3x+7+4x+5$.

Solution:
Solution

	$3x+7+4x+5$
Identify the like terms.	$3x+7+4x+5$
Rearrange the expression, so the like terms are together.	$3x+4x+7+5$
Add the coefficients of the like terms.	$\underbrace{3x+4x}_{7x}+\underbrace{7+5}_{12}$
The original expression is simplified to...	$7x+12$

Note:
Exercise:

Problem: Simplify:

$7x + 9 + 9x + 8$

Solution:

$16x + 17$

Note:
Exercise:

Problem: Simplify:

$5y + 2 + 8y + 4y + 5$

Solution:

$17y + 7$

Example:
Exercise:

Problem: Simplify the expression: $7x^2 + 8x + x^2 + 4x$.

Solution:

Solution

	$7x^2 + 8x + x^2 + 4x$
Identify the like terms.	$7x^2 + 8x + x^2 + 4x$
Rearrange the expression so like terms are together.	$7x^2 + x^2 + 8x + 4x$
Add the coefficients of the like terms.	$8x^2 + 12x$

These are not like terms and cannot be combined. So $8x^2 + 12x$ is in simplest form.

Note:
Exercise:

Problem: Simplify:

$3x^2 + 9x + x^2 + 5x$

Solution:

$4x^2 + 14x$

Note:
Exercise:

Problem: Simplify:

$11y^2 + 8y + y^2 + 7y$

Solution:

$12y^2 + 15y$

Translate Words to Algebraic Expressions

In the previous section, we listed many operation symbols that are used in algebra, and then we translated expressions and equations into word phrases and sentences. Now we'll reverse the process and translate word phrases into algebraic expressions. The symbols and variables we've talked about will help us do that. They are summarized in [link].

Operation	Phrase	Expression
Addition	a plus b the sum of a and b a increased by b b more than a the total of a and b b added to a	$a + b$

Operation	Phrase	Expression
Subtraction	a minus b the difference of a and b b subtracted from a a decreased by b b less than a	$a - b$
Multiplication	a times b the product of a and b	$a \cdot b$, ab, $a(b)$, $(a)(b)$
Division	a divided by b the quotient of a and b the ratio of a and b b divided into a	$a \div b$, a/b, $\frac{a}{b}$, $b\overline{)a}$

Look closely at these phrases using the four operations:

- the sum *of a and b*
- the difference *of a and b*
- the product *of a and b*
- the quotient *of a and b*

Each phrase tells you to operate on two numbers. Look for the words ***of*** and ***and*** to find the numbers.

Example:
Exercise:

Problem: Translate each word phrase into an algebraic expression:

ⓐ the difference of 20 and 4
ⓑ the quotient of $10x$ and 3

Solution:
Solution

ⓐ The key word is *difference*, which tells us the operation is subtraction. Look for the words *of* and *and* to find the numbers to subtract.

$$\text{the difference of } 20 \textit{ and } 4$$
$$20 \text{ minus } 4$$
$$20 - 4$$

ⓑ The key word is *quotient*, which tells us the operation is division.

$$\text{the quotient of } 10x \text{ and } 3$$
$$\text{divide } 10x \text{ by } 3$$
$$10x \div 3$$

This can also be written as $10x/3$ or $\frac{10x}{3}$

Note:
Exercise:

Problem:

Translate the given word phrase into an algebraic expression:

ⓐ the difference of 47 and 41
ⓑ the quotient of $5x$ and 2

Solution:

ⓐ 47 − 41
ⓑ $5x \div 2$

Note:
Exercise:

Problem:

Translate the given word phrase into an algebraic expression:

ⓐ the sum of 17 and 19
ⓑ the product of 7 and x

Solution:

ⓐ 17 + 19
ⓑ 7*x*

How old will you be in eight years? What age is eight more years than your age now? Did you add 8 to your present age? Eight *more than* means eight added to your present age.

How old were you seven years ago? This is seven years less than your age now. You subtract 7 from your present age. Seven *less than* means seven subtracted from your present age.

Example:
Exercise:

Problem: Translate each word phrase into an algebraic expression:

ⓐ Eight more than y
ⓑ Seven less than $9z$

Solution:
Solution

ⓐ The key words are *more than*. They tell us the operation is addition. *More than* means “added to”.

$$\text{Eight more than } y$$
$$\text{Eight added to } y$$
$$y + 8$$

ⓑ The key words are *less than*. They tell us the operation is subtraction. *Less than* means “subtracted from”.

$$\text{Seven less than } 9z$$
$$\text{Seven subtracted from } 9z$$
$$9z - 7$$

Note:
Exercise:

Problem: Translate each word phrase into an algebraic expression:

ⓐ Eleven more than x
ⓑ Fourteen less than $11a$

Solution:

ⓐ $x + 11$
ⓑ $11a - 14$

Note:
Exercise:

Problem: Translate each word phrase into an algebraic expression:

ⓐ19 more than j
ⓑ 21 less than $2x$

Solution:

ⓐ $j + 19$
ⓑ $2x - 21$

Example:
Exercise:

Problem: Translate each word phrase into an algebraic expression:

ⓐ five times the sum of m and n
ⓑ the sum of five times m and n

Solution:
Solution

ⓐ There are two operation words: *times* tells us to multiply and *sum* tells us to add. Because we are multiplying 5 times the sum, we need parentheses around the sum of m and n.

five times the sum of m and n

$$5\,(m+n)$$

ⓑ To take a sum, we look for the words *of* and *and* to see what is being added. Here we are taking the sum *of* five times m and n.

the sum of five times m and n

$$5m+n$$

Notice how the use of parentheses changes the result. In part ⓐ, we add first and in part ⓑ, we multiply first.

Note:
Exercise:

Problem: Translate the word phrase into an algebraic expression:

ⓐ four times the sum of p and q
ⓑ the sum of four times p and q

Solution:

ⓐ $4(p + q)$
ⓐ $4p + q$

Note:
Exercise:

Problem: Translate the word phrase into an algebraic expression:

ⓐ the difference of two times x and 8
ⓑ two times the difference of x and 8

Solution:

ⓐ $2x - 8$
ⓑ $2(x - 8)$

Later in this course, we'll apply our skills in algebra to solving equations. We'll usually start by translating a word phrase to an algebraic expression. We'll need to be clear about what the expression will represent. We'll see how to do this in the next two examples.

Example:
Exercise:

Problem:

The height of a rectangular window is 6 inches less than the width. Let w represent the width of the window. Write an expression for the height of the window.

Solution:
Solution

Write a phrase about the height.	6 less than the width

Substitute w for the width.	6 less than w
Rewrite 'less than' as 'subtracted from'.	6 subtracted from w
Translate the phrase into algebra.	$w - 6$

Note:
Exercise:

Problem:

The length of a rectangle is 5 inches less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Solution:

$w - 5$

Note:
Exercise:

Problem:

The width of a rectangle is 2 meters greater than the length. Let l represent the length of the rectangle. Write an expression for the width of the rectangle.

Solution:

$l + 2$

Example:
Exercise:

Problem:

Blanca has dimes and quarters in her purse. The number of dimes is 2 less than 5 times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Solution:
Solution

Write a phrase about the number of dimes.	two less than five times the number of quarters
Substitute q for the number of quarters.	2 less than five times q
Translate 5 *times* q.	2 less than $5q$
Translate the phrase into algebra.	$5q-2$

Note:
Exercise:

Problem:

Geoffrey has dimes and quarters in his pocket. The number of dimes is seven less than six times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Solution:

$6q - 7$

Note:
Exercise:

Problem:

Lauren has dimes and nickels in her purse. The number of dimes is eight more than four times the number of nickels. Let n represent the number of nickels. Write an expression for the number of dimes.

Solution:

$4n + 8$

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Algebraic Expression Vocabulary

Key Concepts

- **Combine like terms.**

Identify like terms.
Rearrange the expression so like terms are together.
Add the coefficients of the like terms

Practice Makes Perfect

Evaluate Algebraic Expressions

In the following exercises, evaluate the expression for the given value.

Exercise:

Problem: $7x + 8$ when $x = 2$

Solution:

22

Exercise:

Problem: $9x + 7$ when $x = 3$

Exercise:

Problem: $5x - 4$ when $x = 6$

Solution:

26

Exercise:

Problem: $8x - 6$ when $x = 7$

Exercise:

Problem: x^2 when $x = 12$

Solution:

144

Exercise:

Problem: x^3 when $x = 5$

Exercise:

Problem: x^5 when $x = 2$

Solution:

32

Exercise:

Problem: x^4 when $x = 3$

Exercise:

Problem: 3^x when $x = 3$

Solution:

27

Exercise:

Problem: 4^x when $x = 2$

Exercise:

Problem: $x^2 + 3x - 7$ when $x = 4$

Solution:

21

Exercise:

Problem: $x^2 + 5x - 8$ when $x = 6$

Exercise:

Problem: $2x + 4y - 5$ when $x = 7, y = 8$

Solution:

41

Exercise:

Problem: $6x + 3y - 9$ when $x = 6, y = 9$

Exercise:

Problem: $(x - y)^2$ when $x = 10, y = 7$

Solution:

9

Exercise:

Problem: $(x + y)^2$ when $x = 6, y = 9$

Solution:

225

Exercise:

Problem: $a^2 + b^2$ when $a = 3, b = 8$

Solution:

73

Exercise:

Problem: $r^2 - s^2$ when $r = 12, s = 5$

Exercise:

Problem: $2l + 2w$ when $l = 15, w = 12$

Solution:

54

Exercise:

Problem: $2l + 2w$ when $l = 18, w = 14$

Identify Terms, Coefficients, and Like Terms

In the following exercises, list the terms in the given expression.

Exercise:

Problem: $15x^2 + 6x + 2$

Solution:

$15x^2$, $6x$, 2

Exercise:

Problem: $11x^2 + 8x + 5$

Exercise:

Problem: $10y^3 + y + 2$

Solution:

$10y^3$, y, 2

Exercise:

Problem: $9y^3 + y + 5$

In the following exercises, identify the coefficient of the given term.

Exercise:

Problem: $8a$

Solution:

8

Exercise:

Problem: $13m$

Exercise:

Problem: $5r^2$

Solution:

5

Exercise:

Problem: $6x^3$

In the following exercises, identify all sets of like terms.

Exercise:

Problem: $x^3, 8x, 14, 8y, 5, 8x^3$

Solution:

x^3, $8x^3$ and 14, 5

Exercise:

Problem: $6z, 3w^2, 1, 6z^2, 4z, w^2$

Exercise:

Problem: $9a, a^2, 16ab, 16b^2, 4ab, 9b^2$

Solution:

$16ab$ and $4ab$; $16b^2$ and $9b^2$

Exercise:

Problem: $3, 25r^2, 10s, 10r, 4r^2, 3s$

Simplify Expressions by Combining Like Terms

In the following exercises, simplify the given expression by combining like terms.

Exercise:

Problem: $10x + 3x$

Solution:

$13x$

Exercise:

Problem: $15x + 4x$

Exercise:

Problem: $17a + 9a$

Solution:

$26a$

Exercise:

Problem: $18z + 9z$

Exercise:

Problem: $4c + 2c + c$

Solution:

$7c$

Exercise:

Problem: $6y + 4y + y$

Exercise:

Problem: $9x + 3x + 8$

Solution:

$12x + 8$

Exercise:

Problem: $8a + 5a + 9$

Exercise:

Problem: $7u + 2 + 3u + 1$

Solution:

$10u + 3$

Exercise:

Problem: $8d + 6 + 2d + 5$

Exercise:

Problem: $7p + 6 + 5p + 4$

Solution:

$12p + 10$

Exercise:

Problem: $8x + 7 + 4x + 5$

Exercise:

Problem: $10a + 7 + 5a + 2 + 7a + 4$

Solution:

$22a + 13$

Exercise:

Problem: $7c + 4 + 6c + 3 + 9c + 1$

Exercise:

Problem: $3x^2 + 12x + 11 + 14x^2 + 8x + 5$

Solution:

$17x^2 + 20x + 16$

Exercise:

Problem: $5b^2 + 9b + 10 + 2b^2 + 3b + 4$

Translate English Phrases into Algebraic Expressions

In the following exercises, translate the given word phrase into an algebraic expression.

Exercise:

Problem: The sum of 8 and 12

Solution:

$8 + 12$

Exercise:

Problem: The sum of 9 and 1

Exercise:

Problem: The difference of 14 and 9

Solution:

$14 - 9$

Exercise:

Problem: 8 less than 19

Exercise:

Problem: The product of 9 and 7

Solution:

$9 \cdot 7$

Exercise:

Problem: The product of 8 and 7

Exercise:

Problem: The quotient of 36 and 9

Solution:

$36 \div 9$

Exercise:

Problem: The quotient of 42 and 7

Exercise:

Problem: The difference of x and 4

Solution:

$x - 4$

Exercise:

Problem: 3 less than x

Exercise:

Problem: The product of 6 and y

Solution:

$6y$

Exercise:

Problem: The product of 9 and y

Exercise:

Problem: The sum of $8x$ and $3x$

Solution:

$8x + 3x$

Exercise:

Problem: The sum of $13x$ and $3x$

Exercise:

Problem: The quotient of y and 3

Solution:

$\frac{y}{3}$

Exercise:

Problem: The quotient of y and 8

Exercise:

Problem: Eight times the difference of y and nine

Solution:

$8\,(y - 9)$

Exercise:

Problem: Seven times the difference of y and one

Exercise:

Problem: Five times the sum of x and y

Solution:

$5(x+y)$

Exercise:

Problem: Nine times five less than twice x

In the following exercises, write an algebraic expression.

Exercise:

Problem:

Adele bought a skirt and a blouse. The skirt cost $\$15$ more than the blouse. Let b represent the cost of the blouse. Write an expression for the cost of the skirt.

Solution:

$b + 15$

Exercise:

Problem:

Eric has rock and classical CDs in his car. The number of rock CDs is 3 more than the number of classical CDs. Let c represent the number of classical CDs. Write an expression for the number of rock CDs.

Exercise:

Problem:

The number of girls in a second-grade class is 4 less than the number of boys. Let b represent the number of boys. Write an expression for the number of girls.

Solution:

$b-4$

Exercise:

Problem:

Marcella has 6 fewer male cousins than female cousins. Let f represent the number of female cousins. Write an expression for the number of boy cousins.

Exercise:

Problem:

Greg has nickels and pennies in his pocket. The number of pennies is seven less than twice the number of nickels. Let n represent the number of nickels. Write an expression for the number of pennies.

Solution:

$2n-7$

Exercise:

Problem:

Jeannette has $\$5$ and $\$10$ bills in her wallet. The number of fives is three more than six times the number of tens. Let t represent the number of tens. Write an expression for the number of fives.

Everyday Math

In the following exercises, use algebraic expressions to solve the problem.

Exercise:

Problem:

Car insurance Justin's car insurance has a $\$750$ deductible per incident. This means that he pays $\$750$ and his insurance company will pay all costs beyond $\$750$. If Justin files a claim for $\$2,100$, how much will he pay, and how much will his insurance company pay?

Solution:

He will pay \$750. His insurance company will pay \$1350.

Exercise:

Problem:

Home insurance Pam and Armando's home insurance has a $\$2,500$ deductible per incident. This means that they pay $\$2,500$ and their insurance company will pay all costs beyond $\$2,500$. If Pam and Armando file a claim for $\$19,400$, how much will they pay, and how much will their insurance company pay?

Writing Exercises

Exercise:

Problem:

Explain why "the sum of *x* and *y*" is the same as "the sum of *y* and *x*," but "the difference of *x* and *y*" is not the same as "the difference of *y* and *x*." Try substituting two random numbers for x and y to help you explain.

Exercise:

Problem:

Explain the difference between “4 times the sum of x and y” and “the sum of 4 times x and y.”

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate algebraic expressions.			
identify terms, coefficients, and like terms.			
simplify expressions by combining like terms.			
translate word phrases to algebraic expressions.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

term
: A term is a constant or the product of a constant and one or more variables.

coefficient
: The constant that multiplies the variable(s) in a term is called the coefficient.

like terms

Terms that are either constants or have the same variables with the same exponents are like terms.

evaluate

To evaluate an algebraic expression means to find the value of the expression when the variable is replaced by a given number.

Solving Equations Using the Subtraction and Addition Properties of Equality
By the end of this section, you will be able to:

- Determine whether a number is a solution of an equation
- Model the Subtraction Property of Equality
- Solve equations using the Subtraction Property of Equality
- Solve equations using the Addition Property of Equality
- Translate word phrases to algebraic equations
- Translate to an equation and solve

Note:
Before you get started, take this readiness quiz.

1. Evaluate $x+8$ when $x=11$.
   If you missed this problem, review [link].
2. Evaluate $5x-3$ when $x=9$.
   If you missed this problem, review [link].
3. Translate into algebra: the difference of x and 8.
   If you missed this problem, review [link].

When some people hear the word *algebra*, they think of solving equations. The applications of solving equations are limitless and extend to all careers and fields. In this section, we will begin solving equations. We will start by solving basic equations, and then as we proceed through the course we will build up our skills to cover many different forms of equations.

Determine Whether a Number is a Solution of an Equation

Solving an equation is like discovering the answer to a puzzle. An algebraic equation states that two algebraic expressions are equal. To solve an equation is to determine the values of the variable that make the equation a

true statement. Any number that makes the equation true is called a **solution** of the equation. It is the answer to the puzzle!

Note:
Solution of an Equation
A **solution to an equation** is a value of a variable that makes a true statement when substituted into the equation.
The process of finding the solution to an equation is called solving the equation.

To find the solution to an equation means to find the value of the variable that makes the equation true. Can you recognize the solution of $x + 2 = 7$? If you said 5, you're right! We say 5 is a solution to the equation $x + 2 = 7$ because when we substitute 5 for x the resulting statement is true.
Equation:

$$\begin{aligned} x + 2 &= 7 \\ 5 + 2 &\stackrel{?}{=} 7 \\ 7 &= 7\checkmark \end{aligned}$$

Since $5 + 2 = 7$ is a true statement, we know that 5 is indeed a solution to the equation.

The symbol $\stackrel{?}{=}$ asks whether the left side of the equation is equal to the right side. Once we know, we can change to an equal sign $(=)$ or not-equal sign $(\neq)$.

Note:
Determine whether a number is a solution to an equation.

Substitute the number for the variable in the equation.
Simplify the expressions on both sides of the equation.
Determine whether the resulting equation is true.

- If it is true, the number is a solution.
- If it is not true, the number is not a solution.

Example:
Exercise:

Problem:

Determine whether $x = 5$ is a solution of $6x - 17 = 16$.

Solution:
Solution

	$6x - 17 = 16$
Substitute 5 for *x*.	$6 \cdot 5 - 17 \stackrel{?}{=} 16$
Multiply.	$30 - 17 \stackrel{?}{=} 16$

Subtract.	$13 \neq 16$

So $x = 5$ is not a solution to the equation $6x - 17 = 16$.

Note:
Exercise:

Problem: Is $x = 3$ a solution of $4x - 7 = 16$?

Solution:

no

Note:
Exercise:

Problem: Is $x = 2$ a solution of $6x - 2 = 10$?

Solution:

yes

Example:
Exercise:

Problem:

Determine whether $y = 2$ is a solution of $6y - 4 = 5y - 2$.

Solution:
Solution

Here, the variable appears on both sides of the equation. We must substitute 2 for each y.

	$6y - 4 = 5y - 2$
Substitute 2 for y.	$6(2) - 4 \stackrel{?}{=} 5(2) - 2$
Multiply.	$12 - 4 \stackrel{?}{=} 10 - 2$
Subtract.	$8 = 8 \checkmark$

Since $y = 2$ results in a true equation, we know that 2 is a solution to the equation $6y - 4 = 5y - 2$.

Note:
Exercise:

Problem: Is $y = 3$ a solution of $9y - 2 = 8y + 1$?

Solution:

yes

Note:
Exercise:

Problem: Is $y = 4$ a solution of $5y - 3 = 3y + 5$?

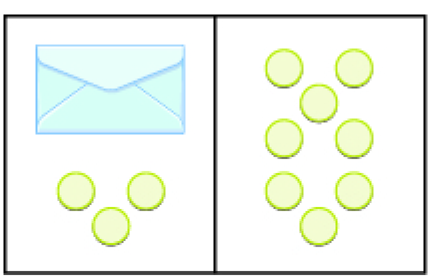
Solution:

yes

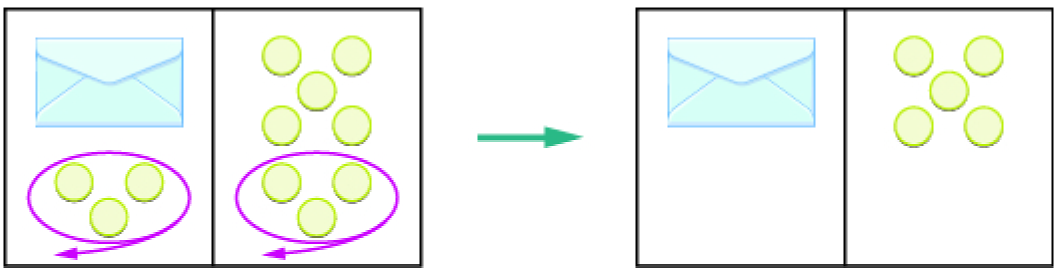
Model the Subtraction Property of Equality

We will use a model to help you understand how the process of solving an equation is like solving a puzzle. An envelope represents the variable – since its contents are unknown – and each counter represents one.

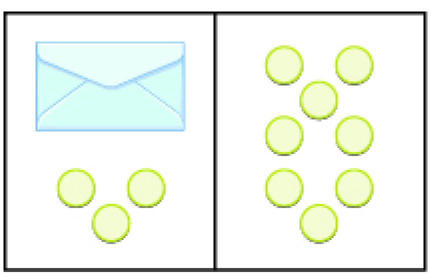
Suppose a desk has an imaginary line dividing it in half. We place three counters and an envelope on the left side of desk, and eight counters on the right side of the desk as in [link]. Both sides of the desk have the same number of counters, but some counters are hidden in the envelope. Can you tell how many counters are in the envelope?



What steps are you taking in your mind to figure out how many counters are in the envelope? Perhaps you are thinking “I need to remove the 3 counters from the left side to get the envelope by itself. Those 3 counters on the left match with 3 on the right, so I can take them away from both sides. That leaves five counters on the right, so there must be 5 counters in the envelope.” [link] shows this process.



What algebraic equation is modeled by this situation? Each side of the desk represents an expression and the center line takes the place of the equal sign. We will call the contents of the envelope x, so the number of counters on the left side of the desk is $x+3$. On the right side of the desk are 8 counters. We are told that $x+3$ is equal to 8 so our equation is $x+3=8$.



Equation:

$$x+3=8$$

Let’s write algebraically the steps we took to discover how many counters were in the envelope.

	$x+3=8$
First, we took away three from each side.	$x+3-3=8-3$
Then we were left with five.	$x=5$

Now let's check our solution. We substitute 5 for x in the original equation and see if we get a true statement.

$$x+3=8$$

$$5+3\stackrel{?}{=}8$$

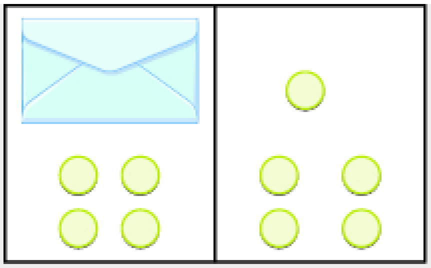
$$8=8\ \checkmark$$

Our solution is correct. Five counters in the envelope plus three more equals eight.

Example:
Exercise:

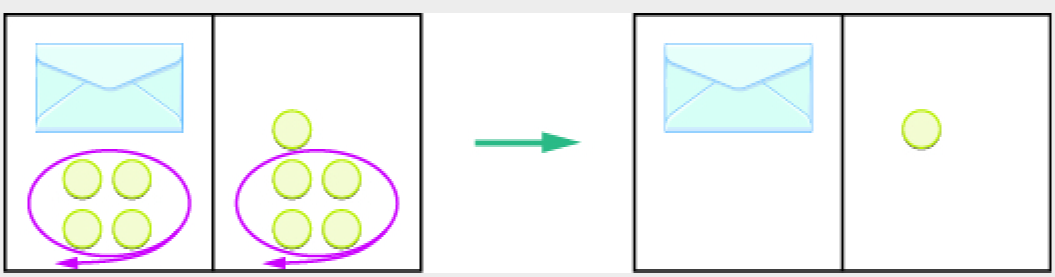
Problem:

Write an equation modeled by the envelopes and counters, and then solve the equation:



Solution:
Solution

On the left, write x for the contents of the envelope, add the 4 counters, so we have $x + 4$.	$x + 4$
On the right, there are 5 counters.	5
The two sides are equal.	$x + 4 = 5$
Solve the equation by subtracting 4 counters from each side.	



We can see that there is one counter in the envelope. This can be shown algebraically as:

$$\begin{aligned} x + 4 &= 5 \\ x + 4 - 4 &= 5 - 4 \\ x &= 1 \end{aligned}$$

Substitute 1 for x in the equation to check.

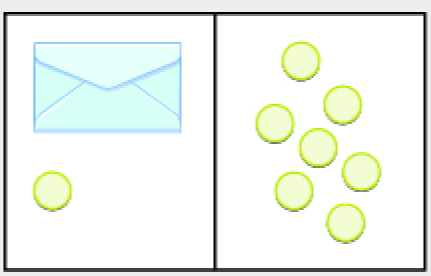
$$\begin{aligned} x + 4 &= 5 \\ 1 + 4 &\stackrel{?}{=} 5 \\ 5 &= 5 \checkmark \end{aligned}$$

Since $x = 1$ makes the statement true, we know that 1 is indeed a solution.

Note:
Exercise:

Problem:

Write the equation modeled by the envelopes and counters, and then solve the equation:



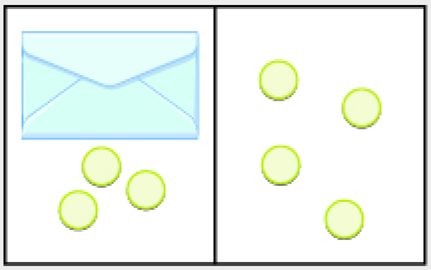
Solution:

$x + 1 = 7$; $x = 6$

Note:
Exercise:

Problem:

Write the equation modeled by the envelopes and counters, and then solve the equation:



Solution:

$x + 3 = 4$; $x = 1$

Solve Equations Using the Subtraction Property of Equality

Our puzzle has given us an idea of what we need to do to solve an equation. The goal is to isolate the variable by itself on one side of the equations. In the previous examples, we used the Subtraction Property of Equality, which states that when we subtract the same quantity from both sides of an equation, we still have equality.

Note:
Subtraction Property of Equality
For any numbers a, b, and c, if
Equation:

$$a = b$$

then
Equation:

$$a - c = b - c$$

Think about twin brothers Andy and Bobby. They are 17 years old. How old was Andy 3 years ago? He was 3 years less than 17, so his age was $17 - 3$, or 14. What about Bobby's age 3 years ago? Of course, he was 14 also. Their ages are equal now, and subtracting the same quantity from both of them resulted in equal ages 3 years ago.
Equation:

$$a = b$$
$$a - 3 = b - 3$$

Note:
Solve an equation using the Subtraction Property of Equality.

Use the Subtraction Property of Equality to isolate the variable.
Simplify the expressions on both sides of the equation.
Check the solution.

Example:
Exercise:

Problem: Solve: $x + 8 = 17$.

Solution:
Solution

We will use the Subtraction Property of Equality to isolate x.

	$x+8=17$
Subtract 8 from both sides.	$x+8-8=17-8$
Simplify.	$x=9$
	$x+8=17$
	$9+8=17$
	$17=17$ ✓

Since $x=9$ makes $x+8=17$ a true statement, we know 9 is the solution to the equation.

Note:
Exercise:

Problem: Solve:

$x + 6 = 19$

Solution:

x = 13

Note:
Exercise:

Problem: Solve:

$x + 9 = 14$

Solution:

x = 5

Example:
Exercise:

Problem: Solve: $100 = y + 74$.

Solution:
Solution

To solve an equation, we must always isolate the variable—it doesn't matter which side it is on. To isolate y, we will subtract 74 from both sides.

	$100 = y + 74$
Subtract 74 from both sides.	$100 - 74 = y + 74 - 74$
Simplify.	$26 = y$
Substitute 26 for y to check. $100 = y + 74$ $100 \stackrel{?}{=} 26 + 74$ $100 = 100\ \checkmark$	

Since $y = 26$ makes $100 = y + 74$ a true statement, we have found the solution to this equation.

Note:
Exercise:

Problem: Solve:

$95 = y + 67$

Solution:

$y = 28$

Note:
Exercise:

Problem: Solve:

$91 = y + 45$

Solution:

$y = 46$

Solve Equations Using the Addition Property of Equality

In all the equations we have solved so far, a number was added to the variable on one side of the equation. We used subtraction to “undo” the addition in order to isolate the variable.

But suppose we have an equation with a number subtracted from the variable, such as $x - 5 = 8$. We want to isolate the variable, so to “undo” the subtraction we will add the number to both sides.

We use the Addition Property of Equality, which says we can add the same number to both sides of the equation without changing the equality. Notice how it mirrors the Subtraction Property of Equality.

Note:
Addition Property of Equality
For any numbers a, b, and c, if
Equation:

$$a = b$$

then
Equation:

$$a+c=b+c$$

Remember the 17-year-old twins, Andy and Bobby? In ten years, Andy's age will still equal Bobby's age. They will both be 27.
Equation:

$$a=b$$
$$a+10=b+10$$

We can add the same number to both sides and still keep the equality.

Note:
Solve an equation using the Addition Property of Equality.

Use the Addition Property of Equality to isolate the variable.
Simplify the expressions on both sides of the equation.
Check the solution.

Example:
Exercise:

Problem: Solve: $x-5=8$.

Solution:
Solution

We will use the Addition Property of Equality to isolate the variable.

	$x - 5 = 8$
Add 5 to both sides.	$x - 5 + 5 = 8 + 5$
Simplify.	$x = 13$
Now we can check. Let $x = 13$.	
$x - 5 = 8$	
$13 - 5 \stackrel{?}{=} 8$	
$8 = 8 \checkmark$	

Note:
Exercise:

Problem: Solve:

$x - 9 = 13$

Solution:

$x = 22$

Note:
Exercise:

Problem: Solve:

$y - 1 = 3$

Solution:

$y = 4$

Example:
Exercise:

Problem: Solve: $27 = a - 16$.

Solution:
Solution

We will add 16 to each side to isolate the variable.

	$27 = a - 16$

Add 16 to each side.	$27 + 16 = a - 16 + 16$
Simplify.	$43 = a$
Now we can check. Let $a = 43$.	$27 = a - 16$
	$27 \stackrel{?}{=} 43 - 16$
	$27 = 27$ ✓

The solution to $27 = a - 16$ is $a = 43$.

Note:
Exercise:

Problem: Solve:

$19 = a - 18$

Solution:

$a = 37$

Note:
Exercise:

Problem: Solve:

$27 = n - 14$

Solution:

$n = 41$

Translate Word Phrases to Algebraic Equations

Remember, an equation has an equal sign between two algebraic expressions. So if we have a sentence that tells us that two phrases are equal, we can translate it into an equation. We look for clue words that mean *equals*. Some words that translate to the equal sign are:

- is equal to
- is the same as
- is
- gives
- was
- will be

It may be helpful to put a box around the *equals* word(s) in the sentence to help you focus separately on each phrase. Then translate each phrase into an expression, and write them on each side of the equal sign.

We will practice translating word sentences into algebraic equations. Some of the sentences will be basic number facts with no variables to solve for. Some sentences will translate into equations with variables. The focus right now is just to translate the words into algebra.

Example:
Exercise:

Problem:

Translate the sentence into an algebraic equation: The sum of 6 and 9 is 15.

Solution:
Solution

The word *is* tells us the equal sign goes between 9 and 15.

Locate the “equals” word(s).	The sum of 6 and 9 $\boxed{\text{is}}$ 15. The sum of 6 and 9 = 15.
Write the = sign.	
Translate the words to the left of the *equals* word into an algebraic expression.	6 + 9 = ___
Translate the words to the right of the *equals* word into an algebraic expression.	6 + 9 = 15

Note:
Exercise:

Problem: Translate the sentence into an algebraic equation:

The sum of 7 and 6 gives 13.

Solution:

$7 + 6 = 13$

Note:
Exercise:

Problem: Translate the sentence into an algebraic equation:

The sum of 8 and 6 is 14.

Solution:

$8 + 6 = 14$

Example:
Exercise:

Problem:

Translate the sentence into an algebraic equation: The product of 8 and 7 is 56.

Solution:
Solution

The location of the word *is* tells us that the equal sign goes between 7 and 56.

Locate the “equals” word(s).	The product of 8 and 7 $\boxed{\text{is}}$ 56.
Write the = sign.	The product of 8 and 7 = 56.
Translate the words to the left of the *equals* word into an algebraic expression.	$8 \cdot 7 = \underline{\quad}$
Translate the words to the right of the *equals* word into an algebraic expression.	$8 \cdot 7 = 56$

Note:
Exercise:

Problem: Translate the sentence into an algebraic equation:

The product of 6 and 9 is 54.

Solution:

$6 \cdot 9 = 54$

Note:
Exercise:

Problem: Translate the sentence into an algebraic equation:

The product of 21 and 3 gives 63.

Solution:

$21 \cdot 3 = 63$

Example:
Exercise:

Problem:

Translate the sentence into an algebraic equation: Twice the difference of x and 3 gives 18.

Solution:
Solution

Locate the “equals” word(s).	Twice the difference of x and 3 $\boxed{\text{gives}}$ 18.
Recognize the key words: *twice; difference of …. and ….*	*Twice* means two times.
Translate.	$\underbrace{\text{Twice}}_{2}$ the $\underbrace{\text{difference of } x \text{ and } 3}_{(x-3)}$ $\underset{=}{\boxed{\text{gives}}}$ $\underbrace{18}_{18}$.

Note:

Exercise:

Problem: Translate the given sentence into an algebraic equation:

Twice the difference of x and 5 gives 30.

Solution:

$2(x - 5) = 30$

Note:
Exercise:

Problem: Translate the given sentence into an algebraic equation:

Twice the difference of y and 4 gives 16.

Solution:

$2(y - 4) = 16$

Translate to an Equation and Solve

Now let's practice translating sentences into algebraic equations and then solving them. We will solve the equations by using the Subtraction and Addition Properties of Equality.

Example:
Exercise:

Problem: Translate and solve: Three more than x is equal to 47.

Solution:
Solution

		Three more than x is equal to 47.
Translate.		$x + 3 = 47$
Subtract 3 from both sides of the equation.		$x + 3 - 3 = 47 - 3$
Simplify.		$x = 44$
We can check. Let $x = 44$.	$x + 3 = 47$	
	$44 + 3 \stackrel{?}{=} 47$	
	$47 = 47$ ✓	

So $x = 44$ is the solution.

Note:
Exercise:

Problem: Translate and solve:

Seven more than x is equal to 37.

Solution:

$x + 7 = 37$; $x = 30$

Note:
Exercise:

Problem: Translate and solve:

Eleven more than y is equal to 28.

Solution:

$y + 11 = 28$; $y = 17$

Example:
Exercise:

Problem: Translate and solve: The difference of y and 14 is 18.

Solution:
Solution

		The difference of y and 14 is 18.
Translate.		$y - 14 = 18$
Add 14 to both sides.		$y - 14 + 14 = 18 + 14$
Simplify.		$y = 32$
We can check. Let $y = 32$.	$y - 14 = 18$	
	$32 - 14 \stackrel{?}{=} 18$	
	$18 = 18$ ✓	

So $y = 32$ is the solution.

Note:
Exercise:

Problem: Translate and solve:

The difference of z and 17 is equal to 37.

Solution:

$z - 17 = 37$; $z = 54$

Note:
Exercise:

Problem: Translate and solve:

The difference of x and 19 is equal to 45.

Solution:

$x - 19 = 45$; $x = 64$

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Solving One Step Equations By Addition and Subtraction

Key Concepts

- **Determine whether a number is a solution to an equation.**

  Substitute the number for the variable in the equation.
  Simplify the expressions on both sides of the equation.
  Determine whether the resulting equation is true. If it is true, the number is a solution.

  If it is not true, the number is not a solution.

- **Subtraction Property of Equality**
    - For any numbers a, b, and c,

if	$a = b$
then	$a - c = b - c$

- **Solve an equation using the Subtraction Property of Equality.**

  Use the Subtraction Property of Equality to isolate the variable.
  Simplify the expressions on both sides of the equation.
  Check the solution.

- **Addition Property of Equality**
    - For any numbers a, b, and c,

if	$a = b$
then	$a + c = b + c$

- **Solve an equation using the Addition Property of Equality.**

  Use the Addition Property of Equality to isolate the variable.
  Simplify the expressions on both sides of the equation.
  Check the solution.

Practice Makes Perfect

Determine Whether a Number is a Solution of an Equation

In the following exercises, determine whether each given value is a solution to the equation.

Exercise:

Problem: $x + 13 = 21$

ⓐ $x = 8$
ⓑ $x = 34$

Solution:

ⓐ yes
ⓑ no

Exercise:

Problem: $y + 18 = 25$

ⓐ $y = 7$
ⓑ $y = 43$

Exercise:

Problem: $m - 4 = 13$

ⓐ $m = 9$
ⓑ $m = 17$

Solution:

ⓐ no

ⓑ yes

Exercise:

Problem: $n - 9 = 6$

ⓐ $n = 3$
ⓑ $n = 15$

Exercise:

Problem: $3p + 6 = 15$

ⓐ $p = 3$
ⓑ $p = 7$

Solution:

ⓐ yes
ⓑ no

Exercise:

Problem: $8q + 4 = 20$

ⓐ $q = 2$
ⓑ $q = 3$

Exercise:

Problem: $18d - 9 = 27$

ⓐ $d = 1$
ⓑ $d = 2$

Solution:

ⓐ no
ⓑ yes

Exercise:

Problem: $24f - 12 = 60$

ⓐ $f = 2$
ⓑ $f = 3$

Exercise:

Problem: $8u - 4 = 4u + 40$

ⓐ $u = 3$
ⓑ $u = 11$

Solution:

ⓐ no
ⓑ yes

Exercise:

Problem: $7v - 3 = 4v + 36$

ⓐ $v = 3$
ⓑ $v = 11$

Exercise:

Problem: $20h - 5 = 15h + 35$

ⓐ $h = 6$
ⓑ $h = 8$

Solution:

ⓐ no
ⓑ yes

Exercise:

Problem: $18k - 3 = 12k + 33$

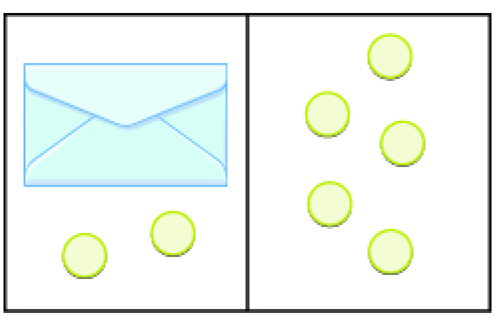
ⓐ $k = 1$
ⓑ $k = 6$

Model the Subtraction Property of Equality

In the following exercises, write the equation modeled by the envelopes and counters and then solve using the subtraction property of equality.

Exercise:

Problem:

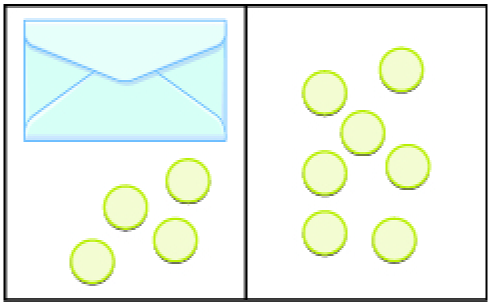


Solution:

$x + 2 = 5$; $x = 3$

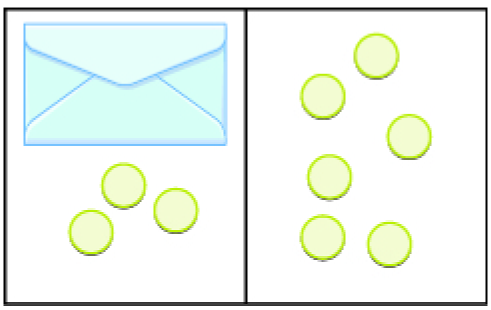
Exercise:

Problem:



Exercise:

Problem:

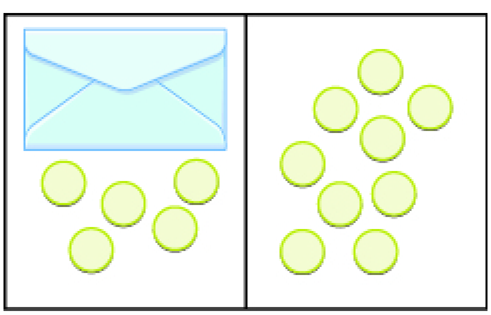


Solution:

$x + 3 = 6$; $x = 3$

Exercise:

Problem:



Solve Equations using the Subtraction Property of Equality

In the following exercises, solve each equation using the subtraction property of equality.

Exercise:

Problem: $a+2=18$

Solution:

$a = 16$

Exercise:

Problem: $b+5=13$

Exercise:

Problem: $p+18=23$

Solution:

$p = 5$

Exercise:

Problem: $q+14=31$

Exercise:

Problem: $r+76=100$

Solution:

$r = 24$

Exercise:

Problem: $s + 62 = 95$

Exercise:

Problem: $16 = x + 9$

Solution:

$x = 7$

Exercise:

Problem: $17 = y + 6$

Exercise:

Problem: $93 = p + 24$

Solution:

$p = 69$

Exercise:

Problem: $116 = q + 79$

Exercise:

Problem: $465 = d + 398$

Solution:

$d = 67$

Exercise:

Problem: $932 = c + 641$

Solve Equations using the Addition Property of Equality

In the following exercises, solve each equation using the addition property of equality.

Exercise:

Problem: $y - 3 = 19$

Solution:

$y = 22$

Exercise:

Problem: $x - 4 = 12$

Exercise:

Problem: $u - 6 = 24$

Solution:

$u = 30$

Exercise:

Problem: $v - 7 = 35$

Exercise:

Problem: $f - 55 = 123$

Solution:

$f = 178$

Exercise:

Problem: $g - 39 = 117$

Exercise:

Problem: $19 = n - 13$

Solution:

$n = 32$

Exercise:

Problem: $18 = m - 15$

Exercise:

Problem: $10 = p - 38$

Solution:

$p = 48$

Exercise:

Problem: $18 = q - 72$

Exercise:

Problem: $268 = y - 199$

Solution:

$y = 467$

Exercise:

Problem: $204 = z - 149$

Translate Word Phrase to Algebraic Equations

In the following exercises, translate the given sentence into an algebraic equation.

Exercise:

Problem: The sum of 8 and 9 is equal to 17.

Solution:

$8 + 9 = 17$

Exercise:

Problem: The sum of 7 and 9 is equal to 16.

Exercise:

Problem: The difference of 23 and 19 is equal to 4.

Solution:

$23 - 19 = 4$

Exercise:

Problem: The difference of 29 and 12 is equal to 17.

Exercise:

Problem: The product of 3 and 9 is equal to 27.

Solution:

$3 \cdot 9 = 27$

Exercise:

Problem: The product of 6 and 8 is equal to 48.

Exercise:

Problem: The quotient of 54 and 6 is equal to 9.

Solution:

$54 \div 6 = 9$

Exercise:

Problem: The quotient of 42 and 7 is equal to 6.

Exercise:

Problem: Twice the difference of n and 10 gives 52.

Solution:

$2(n - 10) = 52$

Exercise:

Problem: Twice the difference of m and 14 gives 64.

Exercise:

Problem: The sum of three times y and 10 is 100.

Solution:

$3y + 10 = 100$

Exercise:

Problem: The sum of eight times x and 4 is 68.

Translate to an Equation and Solve

In the following exercises, translate the given sentence into an algebraic equation and then solve it.

Exercise:

Problem: Five more than p is equal to 21.

Solution:

$p + 5 = 21; p = 16$

Exercise:

Problem: Nine more than q is equal to 40.

Exercise:

Problem: The sum of r and 18 is 73.

Solution:

$r + 18 = 73; r = 55$

Exercise:

Problem: The sum of s and 13 is 68.

Exercise:

Problem: The difference of d and 30 is equal to 52.

Solution:

$d - 30 = 52; d = 82$

Exercise:

Problem: The difference of c and 25 is equal to 75.

Exercise:

Problem: 12 less than u is 89.

Solution:

$u - 12 = 89$; $u = 101$

Exercise:

Problem: 19 less than w is 56.

Exercise:

Problem: 325 less than c gives 799.

Solution:

$c - 325 = 799$; $c = 1124$

Exercise:

Problem: 299 less than d gives 850.

Everyday Math

Exercise:

Problem:

Insurance Vince's car insurance has a $\$500$ deductible. Find the amount the insurance company will pay, p, for an $\$1800$ claim by solving the equation $500 + p = 1800$.

Solution:

$1300

Exercise:

Problem:

Insurance Marta's homeowner's insurance policy has a $\$750$ deductible. The insurance company paid $\$5800$ to repair damages caused by a storm. Find the total cost of the storm damage, d, by solving the equation $d - 750 = 5800$.

Exercise:

Problem:

Sale purchase Arthur bought a suit that was on sale for $\$120$ off. He paid $\$340$ for the suit. Find the original price, p, of the suit by solving the equation $p - 120 = 340$.

Solution:

$460

Exercise:

Problem:

Sale purchase Rita bought a sofa that was on sale for $\$1299$. She paid a total of $\$1409$, including sales tax. Find the amount of the sales tax, t, by solving the equation $1299 + t = 1409$.

Writing Exercises

Exercise:

Problem:

Is $x = 1$ a solution to the equation $8x - 2 = 16 - 6x$? How do you know?

Exercise:

Problem:

Write the equation $y - 5 = 21$ in words. Then make up a word problem for this equation.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether a number is a solution of an equation.			
model the subtraction property of equality.			
solve equations using the subtraction property of equality.			
solve equations using the addition property of equality.			
translate word phrases to algebraic equations.			
translate to an equation and solve.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

solution of an equation
: A solution to an equation is a value of a variable that makes a true statement when substituted into the equation. The process of finding the solution to an equation is called solving the equation.

Find Multiples and Factors
By the end of this section, you will be able to:

- Identify multiples of numbers
- Use common divisibility tests
- Find all the factors of a number
- Identify prime and composite numbers

Note:
Before you get started, take this readiness quiz.

1. Which of the following numbers are counting numbers (natural numbers)?
   $0, 4, 215$
   If you missed this problem, review [link].
2. Find the sum of $3, 5$, and 7.
   If you missed the problem, review [link].

Identify Multiples of Numbers

Annie is counting the shoes in her closet. The shoes are matched in pairs, so she doesn't have to count each one. She counts by twos: $2, 4, 6, 8, 10, 12$. She has 12 shoes in her closet.

The numbers $2, 4, 6, 8, 10, 12$ are called multiples of 2. Multiples of 2 can be written as the product of a counting number and 2. The first six multiples of 2 are given below.
Equation:

$$\begin{array}{c}1\cdot 2=2\\ 2\cdot 2=4\\ 3\cdot 2=6\\ 4\cdot 2=8\\ 5\cdot 2=10\\ 6\cdot 2=12\end{array}$$

A **multiple of a number** is the product of the number and a counting number. So a multiple of 3 would be the product of a counting number and 3. Below are the first six multiples of 3.
Equation:

$$\begin{array}{c}1\cdot 3=3\\ 2\cdot 3=6\\ 3\cdot 3=9\\ 4\cdot 3=12\\ 5\cdot 3=15\\ 6\cdot 3=18\end{array}$$

We can find the multiples of any number by continuing this process. [link] shows the multiples of 2 through 9 for the first twelve counting numbers.

Counting Number	1	2	3	4	5	6	7	8	9	10	11	12
Multiples of 2	2	4	6	8	10	12	14	16	18	20	22	24
Multiples of 3	3	6	9	12	15	18	21	24	27	30	33	36
Multiples of 4	4	8	12	16	20	24	28	32	36	40	44	48
Multiples of 5	5	10	15	20	25	30	35	40	45	50	55	60
Multiples of 6	6	12	18	24	30	36	42	48	54	60	66	72
Multiples of 7	7	14	21	28	35	42	49	56	63	70	77	84
Multiples of 8	8	16	24	32	40	48	56	64	72	80	88	96
Multiples of 9	9	18	27	36	45	54	63	72	81	90	99	108

Note:
Multiple of a Number
A number is a multiple of n if it is the product of a counting number and n.

Recognizing the patterns for multiples of $2, 5, 10,$ and 3 will be helpful to you as you continue in this course.

[link] shows the counting numbers from 1 to 50. Multiples of 2 are highlighted. Do you notice a pattern?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Multiples of 2 between 1 and 50

The last digit of each highlighted number in [link] is either $0, 2, 4, 6,$ or 8. This is true for the product of 2 and any counting number. So, to tell if any number is a multiple of 2 look at the last digit. If it is $0, 2, 4, 6,$ or 8, then the number is a multiple of 2.

Example:
Exercise:

Problem: Determine whether each of the following is a multiple of 2:

ⓐ 489
ⓑ $3,714$

Solution:
Solution

ⓐ	
Is 489 a multiple of 2?	
Is the last digit 0, 2, 4, 6, or 8?	No.
	489 is not a multiple of 2.

ⓑ	
Is 3,714 a multiple of 2?	
Is the last digit 0, 2, 4, 6, or 8?	Yes.
	3,714 is a multiple of 2.

Note:
Exercise:

Problem: Determine whether each number is a multiple of 2:

ⓐ 678
ⓑ 21,493

Solution:

ⓐ yes
ⓑ no

Note:
Exercise:

Problem: Determine whether each number is a multiple of 2:

ⓐ 979
ⓑ 17,780

Solution:

ⓐ no
ⓑ yes

Now let's look at multiples of 5. [link] highlights all of the multiples of 5 between 1 and 50. What do you notice about the multiples of 5?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Multiples of 5 between 1 and 50

All multiples of 5 end with either 5 or 0. Just like we identify multiples of 2 by looking at the last digit, we can identify multiples of 5 by looking at the last digit.

Example:
Exercise:

Problem: Determine whether each of the following is a multiple of 5:

ⓐ 579
ⓑ 880

Solution:
Solution

ⓐ	
Is 579 a multiple of 5?	
Is the last digit 5 or 0?	No.
	579 is not a multiple of 5.

ⓑ	
Is 880 a multiple of 5?	
Is the last digit 5 or 0?	Yes.
	880 is a multiple of 5.

Note:
Exercise:

Problem: Determine whether each number is a multiple of 5.

ⓐ 675
ⓑ 1,578

Solution:

ⓐ yes
ⓑ no

Note:
Exercise:

Problem: Determine whether each number is a multiple of 5.

ⓐ 421
ⓑ 2,690

Solution:

ⓐ no
ⓑ yes

[link] highlights the multiples of 10 between 1 and 50. All multiples of 10 all end with a zero.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Multiples of 10 between 1 and 50

Example:
Exercise:

Problem: Determine whether each of the following is a multiple of 10:

ⓐ 425
ⓑ 350

Solution:
Solution

ⓐ	
Is 425 a multiple of 10?	
Is the last digit zero?	No.
	425 is not a multiple of 10.

ⓑ	
Is 350 a multiple of 10?	
Is the last digit zero?	Yes.
	350 is a multiple of 10.

Note:
Exercise:

Problem: Determine whether each number is a multiple of 10:

ⓐ 179
ⓑ 3,540

Solution:

ⓐ no
ⓑ yes

Note:
Exercise:

Problem: Determine whether each number is a multiple of 10:

ⓐ 110
ⓑ $7{,}595$

Solution:

ⓐ yes
ⓑ no

[link] highlights multiples of 3. The pattern for multiples of 3 is not as obvious as the patterns for multiples of 2, 5, and 10.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Multiples of 3 between 1 and 50

Unlike the other patterns we've examined so far, this pattern does not involve the last digit. The pattern for multiples of 3 is based on the sum of the digits. If the sum of the digits of a number is a multiple of 3, then the number itself is a multiple of 3. See [link].

Multiple of 3	3	6	9	12	15	18	21	24
Sum of digits	3	6	9	$1+2$ 3	$1+5$ 6	$1+8$ 9	$2+1$ 3	$2+4$ 6

Consider the number 42. The digits are 4 and 2, and their sum is $4+2=6$. Since 6 is a multiple of 3, we know that 42 is also a multiple of 3.

Example:
Exercise:

Problem: Determine whether each of the given numbers is a multiple of 3:

ⓐ 645
ⓑ $10,519$

Solution:
Solution

ⓐ Is 645 a multiple of 3?

Find the sum of the digits.	$6+4+5=15$
Is 15 a multiple of 3?	Yes.
If we're not sure, we could add its digits to find out. We can check it by dividing 645 by 3.	$645 \div 3$
The quotient is 215.	$3 \cdot 215 = 645$

ⓑ Is $10,519$ a multiple of 3?

Find the sum of the digits.	$1+0+5+1+9=16$
Is 16 a multiple of 3?	No.
So 10,519 is not a multiple of 3 either..	$645 \div 3$
We can check this by dividing by 10,519 by 3.	$3\overline{)10,519}$ with quotient $3,506\text{R}1$

When we divide $10,519$ by 3, we do not get a counting number, so $10,519$ is not the product of a counting number and 3. It is not a multiple of 3.

Note:
Exercise:

Problem: Determine whether each number is a multiple of 3:

ⓐ 954
ⓑ 3,742

Solution:

ⓐ yes
ⓑ no

Note:
Exercise:

Problem: Determine whether each number is a multiple of 3:

ⓐ 643
ⓑ 8,379

Solution:

ⓐ no
ⓑ yes

Look back at the charts where you highlighted the multiples of 2, of 5, and of 10. Notice that the multiples of 10 are the numbers that are multiples of both 2 and 5. That is because $10 = 2 \cdot 5$. Likewise, since $6 = 2 \cdot 3$, the multiples of 6 are the numbers that are multiples of both 2 and 3.

Use Common Divisibility Tests

Another way to say that 375 is a multiple of 5 is to say that 375 is divisible by 5. In fact, $375 \div 5$ is 75, so 375 is $5 \cdot 75$. Notice in [link] that 10,519 is not a multiple 3. When we divided 10,519 by 3 we did not get a counting number, so 10,519 is not divisible by 3.

Note:
Divisibility
If a number m is a multiple of n, then we say that m is divisible by n.

Since multiplication and division are inverse operations, the patterns of multiples that we found can be used as divisibility tests. [link] summarizes divisibility tests for some of the counting numbers between one and ten.

Divisibility Tests

Divisibility Tests A number is divisible by	
A number is divisible by	
2	if the last digit is $0, 2, 4, 6,$ or 8
3	if the sum of the digits is divisible by 3
5	if the last digit is 5 or 0
6	if divisible by both 2 and 3
10	if the last digit is 0

Example:
Exercise:

Problem: Determine whether 1,290 is divisible by 2, 3, 5, and 10.

Solution:
Solution

[link] applies the divisibility tests to 1,290. In the far right column, we check the results of the divisibility tests by seeing if the quotient is a whole number.

Divisible by…?	**Test**	**Divisible?**	**Check**
2	Is last digit $0, 2, 4, 6,$ or 8? *Yes.*	yes	$1290 \div 2 = 645$
3	Is sum of digits divisible by 3? $1+2+9+0=12$ *Yes.*	yes	$1290 \div 3 = 430$
5	Is last digit 5 or 0? *Yes.*	yes	$1290 \div 5 = 258$
10	Is last digit 0? *Yes.*	yes	$1290 \div 10 = 129$

Thus, 1,290 is divisible by 2, 3, 5, and 10.

Note:
Exercise:

Problem: Determine whether the given number is divisible by 2, 3, 5, and 10.

6240

Solution:

Divisible by 2, 3, 5, and 10

Note:
Exercise:

Problem: Determine whether the given number is divisible by $2, 3, 5,$ and $10.$

7248

Solution:

Divisible by 2 and 3, not 5 or 10.

Example:
Exercise:

Problem: Determine whether $5{,}625$ is divisible by $2, 3, 5,$ and $10.$

Solution:
Solution

[link] applies the divisibility tests to $5{,}625$ and tests the results by finding the quotients.

Divisible by…?	**Test**	**Divisible?**	**Check**
2	Is last digit $0,\ 2,\ 4,\ 6,$ or 8? *No.*	no	$5625 \div 2 = 2812.5$
3	Is sum of digits divisible by 3? $5 + 6 + 2 + 5 = 18$ *Yes.*	yes	$5625 \div 3 = 1875$
5	Is last digit is 5 or 0? *Yes.*	yes	$5625 \div 5 = 1125$
10	Is last digit 0? *No.*	no	$5625 \div 10 = 562.5$

Thus, $5{,}625$ is divisible by 3 and $5,$ but not $2,$ or $10.$

Note:
Exercise:

Problem: Determine whether the given number is divisible by $2, 3, 5,$ and $10.$

4962

Solution:

Divisible by 2, 3, not 5 or 10.

Note:
Exercise:

Problem: Determine whether the given number is divisible by $2, 3, 5$, and 10.

3765

Solution:

Divisible by 3 and 5.

Find All the Factors of a Number

There are often several ways to talk about the same idea. So far, we've seen that if m is a multiple of n, we can say that m is divisible by n. We know that 72 is the product of 8 and 9, so we can say 72 is a multiple of 8 and 72 is a multiple of 9. We can also say 72 is divisible by 8 and by 9. Another way to talk about this is to say that 8 and 9 are factors of 72. When we write $72 = 8 \cdot 9$ we can say that we have factored 72.

$$\underbrace{8 \cdot 9}_{\textit{factors}} = \underbrace{72}_{\textit{product}}$$

Note:
Factors
If $a \cdot b = m$, then a and b are factors of m, and m is the product of a and b.

In algebra, it can be useful to determine all of the factors of a number. This is called factoring a number, and it can help us solve many kinds of problems.

For example, suppose a choreographer is planning a dance for a ballet recital. There are 24 dancers, and for a certain scene, the choreographer wants to arrange the dancers in groups of equal sizes on stage.

In how many ways can the dancers be put into groups of equal size? Answering this question is the same as identifying the factors of 24. [link] summarizes the different ways that the choreographer can arrange the dancers.

Number of Groups	Dancers per Group	Total Dancers
1	24	$1 \cdot 24 = 24$

Number of Groups	Dancers per Group	Total Dancers
2	12	$2 \cdot 12 = 24$
3	8	$3 \cdot 8 = 24$
4	6	$4 \cdot 6 = 24$
6	4	$6 \cdot 4 = 24$
8	3	$8 \cdot 3 = 24$
12	2	$12 \cdot 2 = 24$
24	1	$24 \cdot 1 = 24$

What patterns do you see in [link]? Did you notice that the number of groups times the number of dancers per group is always 24? This makes sense, since there are always 24 dancers.

You may notice another pattern if you look carefully at the first two columns. These two columns contain the exact same set of numbers—but in reverse order. They are mirrors of one another, and in fact, both columns list all of the factors of 24, which are:

Equation:

$$1, 2, 3, 4, 6, 8, 12, 24$$

We can find all the factors of any counting number by systematically dividing the number by each counting number, starting with 1. If the quotient is also a counting number, then the divisor and the quotient are factors of the number. We can stop when the quotient becomes smaller than the divisor.

Note:
Find all the factors of a counting number.

Divide the number by each of the counting numbers, in order, until the quotient is smaller than the divisor.

- If the quotient is a counting number, the divisor and quotient are a pair of factors.
- If the quotient is not a counting number, the divisor is not a factor.

List all the factor pairs.
Write all thefactorsin order from smallest to largest.

Example:
Exercise:

Problem: Find all the factors of 72.

Solution:
Solution

Divide 72 by each of the counting numbers starting with 1. If the quotient is a whole number, the divisor and quotient are a pair of factors.

Dividend	Divisor	Quotient	Factors
72	1	72	1, 72
72	2	36	2, 36
72	3	24	3, 24
72	4	18	4, 18
72	5	14.4	–
72	6	12	6, 12
72	7	~10.29	–
72	8	9	8, 9

The next line would have a divisor of 9 and a quotient of 8. The quotient would be smaller than the divisor, so we stop. If we continued, we would end up only listing the same factors again in reverse order. Listing all the factors from smallest to greatest, we have

$1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, \text{and } 72$

Note:
Exercise:

Problem: Find all the factors of the given number:

96

Solution:

1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96

Note:
Exercise:

Problem: Find all the factors of the given number:

80

Solution:

1, 2, 4, 5, 8, 10, 16, 20, 40, 80

Identify Prime and Composite Numbers

Some numbers, like 72, have many factors. Other numbers, such as 7, have only two factors: 1 and the number. A number with only two factors is called a **prime number**. A number with more than two factors is called a **composite number**. The number 1 is neither prime nor composite. It has only one factor, itself.

Note:
Prime Numbers and Composite Numbers

A prime number is a counting number greater than 1 whose only factors are 1 and itself.
A composite number is a counting number that is not prime.

[link] lists the counting numbers from 2 through 20 along with their factors. The highlighted numbers are prime, since each has only two factors.

Number	Factors	Prime or Composite?		Number	Factor	Prime or Composite?
2	1,2	Prime		12	1,2,3,4,6,12	Composite
3	1,3	Prime		13	1,13	Prime
4	1,2,4	Composite		14	1,2,7,14	Composite
5	1,5	Prime		15	1,3,5,15	Composite
6	1,2,3,6	Composite		16	1,2,4,8,16	Composite
7	1,7	Prime		17	1,17	Prime
8	1,2,4,8	Composite		18	1,2,3,6,9,18	Composite
9	1,3,9	Composite		19	1,19	Prime
10	1,2,5,10	Composite		20	1,2,4,5,10,20	Composite
11	1,11	Prime				

Factors of the counting numbers from 2 through 20, with prime numbers highlighted

The prime numbers less than 20 are $2, 3, 5, 7, 11, 13, 17, \text{and } 19$. There are many larger prime numbers too. In order to determine whether a number is prime or composite, we need to see if the number has any factors other than 1 and itself. To do this, we can test each of the smaller prime numbers in order to see if it is a factor of the number. If none of the prime numbers are factors, then that number is also prime.

Note:
Determine if a number is prime.

Test each of the primes, in order, to see if it is a factor of the number.
Start with 2 and stop when the quotient is smaller than the divisor or when a prime factor is found.
If the number has aprime factor, then it is acomposite number. If it has no prime factors, then the number is prime.

Example:
Exercise:

Problem: Identify each number as prime or composite:

ⓐ 83
ⓑ 77

Solution:
Solution

ⓐ Test each prime, in order, to see if it is a factor of 83, starting with 2, as shown. We will stop when the quotient is smaller than the divisor.

Prime	**Test**	**Factor of** 83?
2	Last digit of 83 is not $0, 2, 4, 6,$ or 8.	No.
3	$8 + 3 = 11$, and 11 is not divisible by 3.	No.
5	The last digit of 83 is not 5 or 0.	No.
7	$83 \div 7 = 11.857\ldots$.	No.
11	$83 \div 11 = 7.545\ldots$	No.

We can stop when we get to 11 because the quotient $(7.545\ldots)$ is less than the divisor.

We did not find any prime numbers that are factors of 83, so we know 83 is prime.

ⓑ Test each prime, in order, to see if it is a factor of 77.

Prime	**Test**	**Factor of** 77?
2	Last digit is not $0, 2, 4, 6,$ or 8.	No.
3	$7 + 7 = 14$, and 14 is not divisible by 3.	No.
5	the last digit is not 5 or 0.	No.
7	$77 \div 11 = 7$	Yes.

Since 77 is divisible by 7, we know it is not a prime number. It is composite.

Note:
Exercise:

Problem: Identify the number as prime or composite:

91

Solution:

composite

Note:
Exercise:

Problem: Identify the number as prime or composite:

137

Solution:

prime

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Divisibility Rules
- Factors
- Ex 1: Determine Factors of a Number
- Ex 2: Determine Factors of a Number
- Ex 3: Determine Factors of a Number

Key Concepts

Divisibility Tests	
A number is divisible by	
2	if the last digit is **0, 2, 4, 6,** or **8**
3	if the sum of the digits is divisible by **3**
5	if the last digit is **5** or **0**
6	if divisible by both **2** and **3**
10	if the last digit is **0**

- **Factors** If $a \cdot b = m$, then a and b are factors of m, and m is the product of a and b.
- **Find all the factors of a counting number.**

  Divide the number by each of the counting numbers, in order, until the quotient is smaller than the divisor.

  a. If the quotient is a counting number, the divisor and quotient are a pair of factors.

b. If the quotient is not a counting number, the divisor is not a factor.

List all the factor pairs.
Write all the factors in order from smallest to largest.

- **Determine if a number is prime.**

  Test each of the primes, in order, to see if it is a factor of the number.
  Start with 2 and stop when the quotient is smaller than the divisor or when a prime factor is found.
  If the number has a prime factor, then it is a composite number. If it has no prime factors, then the number is prime.

Practice Makes Perfect

Identify Multiples of Numbers

In the following exercises, list all the multiples less than 50 for the given number.

Exercise:

Problem: 2

Solution:

2, 4, 6, 8, 10 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48

Exercise:

Problem: 3

Exercise:

Problem: 4

Solution:

4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48

Exercise:

Problem: 5

Exercise:

Problem: 6

Solution:

6, 12, 18, 24, 30, 36, 42, 48

Exercise:

Problem: 7

Exercise:

Problem: 8

Solution:

8, 16, 24, 32, 40, 48

Exercise:

Problem: 9

Exercise:

Problem: 10

Solution:

10, 20, 30, 40

Exercise:

Problem: 12

Use Common Divisibility Tests

In the following exercises, use the divisibility tests to determine whether each number is divisible by $2, 3, 4, 5, 6,$ and $10.$

Exercise:

Problem: 84

Solution:

Divisible by 2, 3, 4, 6

Exercise:

Problem: 96

Exercise:

Problem: 75

Solution:

Divisible by 3, 5

Exercise:

Problem: 78

Exercise:

Problem: 168

Solution:

Divisible by 2, 3, 4, 6

Exercise:

Problem: 264

Exercise:

Problem: 900

Solution:

Divisible by 2, 3, 4, 5, 6, 10

Exercise:

Problem: 800

Exercise:

Problem: 896

Solution:

Divisible by 2, 4

Exercise:

Problem: 942

Exercise:

Problem: 375

Solution:

Divisible by 3, 5

Exercise:

Problem: 750

Exercise:

Problem: 350

Solution:

Divisible by 2, 5, 10

Exercise:

Problem: 550

Exercise:

Problem: 1430

Solution:

Divisible by 2, 5, 10

Exercise:

Problem: 1080

Exercise:

Problem: 22,335

Solution:

Divisible by 3, 5

Exercise:

Problem: 39,075

Find All the Factors of a Number

In the following exercises, find all the factors of the given number.

Exercise:

Problem: 36

Solution:

1, 2, 3, 4, 6, 9, 12, 18, 36

Exercise:

Problem: 42

Exercise:

Problem: 60

Solution:

1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60

Exercise:

Problem: 48

Exercise:

Problem: 144

Solution:

1, 2, 3, 4, 6, 8, 12, 18, 24, 36, 48, 72,144

Exercise:

Problem: 200

Exercise:

Problem: 588

Solution:

1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 49, 84, 98, 147, 196, 294, 588

Exercise:

Problem: 576

Identify Prime and Composite Numbers

In the following exercises, determine if the given number is prime or composite.

Exercise:

Problem: 43

Solution:

prime

Exercise:

Problem: 67

Exercise:

Problem: 39

Solution:

composite

Exercise:

Problem: 53

Exercise:

Problem: 71

Solution:

prime

Exercise:

Problem: 119

Exercise:

Problem: 481

Solution:

composite

Exercise:

Problem: 221

Exercise:

Problem: 209

Solution:

composite

Exercise:

Problem: 359

Exercise:

Problem: 667

Solution:

composite

Exercise:

Problem: 1771

Everyday Math

Exercise:

Problem:

Banking Frank's grandmother gave him $100 at his high school graduation. Instead of spending it, Frank opened a bank account. Every week, he added $15 to the account. The table shows how much money Frank had put in the account by the end of each week. Complete the table by filling in the blanks.

Weeks after graduation	Total number of dollars Frank put in the account	Simplified Total
0	100	100
1	$100+15$	115
2	$100+15\cdot 2$	130
3	$100+15\cdot 3$	
4	$100+15\cdot [\,]$	

Weeks after graduation	Total number of dollars Frank put in the account	Simplified Total
5	$100 + [\,]$	
6		
20		
x		

Solution:

Weeks after graduation	Total number of dollars Frank put in the account	Simplified Total
0	100	100
1	$100 + 15$	115
2	$100 + 15 \cdot 2$	130
3	$100 + 15 \cdot 3$	145
4	$100 + 15 \cdot 4$	160
5	$100 + 15 \cdot 5$	175
6	$100 + 15 \cdot 6$	190
20	$100 + 15 \cdot 20$	400
x	$100 + 15 \cdot x$	$100 + 15x$

Exercise:

Problem:

Banking In March, Gina opened a Christmas club savings account at her bank. She deposited $75 to open the account. Every week, she added $20 to the account. The table shows how much money Gina had put in the account by the end of each week. Complete the table by filling in the blanks.

Weeks after opening the account	Total number of dollars Gina put in the account	Simplified Total
0	75	75
1	$75 + 20$	95
2	$75 + 20 \cdot 2$	115
3	$75 + 20 \cdot 3$	
4	$75 + 20 \cdot [\,]$	
5	$75 + [\,]$	

Weeks after opening the account	Total number of dollars Gina put in the account	Simplified Total
6		
20		
x		

Writing Exercises

Exercise:

Problem: If a number is divisible by 2 and by 3, why is it also divisible by 6?

Exercise:

Problem: What is the difference between prime numbers and composite numbers?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
identify multiples of numbers.			
use common divisibility tests.			
find all the factors of a number.			
identify prime and composite numbers.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

multiple of a number
: A number is a multiple of n if it is the product of a counting number and n.

divisibility
: If a number m is a multiple of n, then we say that m is divisible by n.

prime number
: A prime number is a counting number greater than 1 whose only factors are 1 and itself.

composite number
: A composite number is a counting number that is not prime.

Prime Factorization and the Least Common Multiple
By the end of this section, you will be able to:

- Find the prime factorization of a composite number
- Find the least common multiple (LCM) of two numbers

Note:
Before you get started, take this readiness quiz.

1. Is 810 divisible by $2, 3, 5, 6,$ or 10?
   If you missed this problem, review [link].
2. Is 127 prime or composite?
   If you missed this problem, review [link].
3. Write $2 \cdot 2 \cdot 2 \cdot 2$ in exponential notation.
   If you missed this problem, review [link].

Find the Prime Factorization of a Composite Number

In the previous section, we found the factors of a number. Prime numbers have only two factors, the number 1 and the prime number itself. Composite numbers have more than two factors, and every composite number can be written as a unique product of primes. This is called the **prime factorization** of a number. When we write the prime factorization of a number, we are rewriting the number as a product of primes. Finding the prime factorization of a composite number will help you later in this course.

Note:
Prime Factorization
The prime factorization of a number is the product of prime numbers that equals the number.

You may want to refer to the following list of prime numbers less than 50 as you work through this section.

Equation:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$$

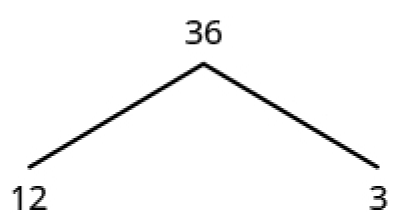
Prime Factorization Using the Factor Tree Method

One way to find the prime factorization of a number is to make a factor tree. We start by writing the number, and then writing it as the product of two factors. We write the factors below the number and connect them to the number with a small line segment—a "branch" of the factor tree.

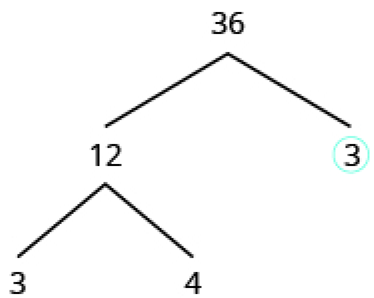
If a factor is prime, we circle it (like a bud on a tree), and do not factor that "branch" any further. If a factor is not prime, we repeat this process, writing it as the product of two factors and adding new branches to the tree.

We continue until all the branches end with a prime. When the factor tree is complete, the circled primes give us the prime factorization.

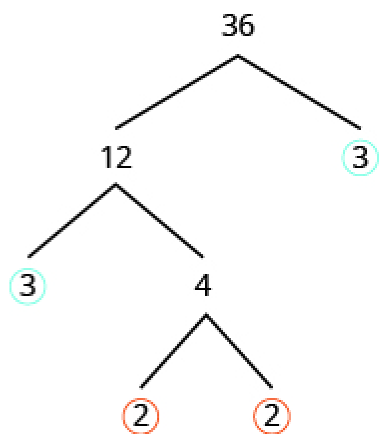
For example, let's find the prime factorization of 36. We can start with any factor pair such as 3 and 12. We write 3 and 12 below 36 with branches connecting them.



The factor 3 is prime, so we circle it. The factor 12 is composite, so we need to find its factors. Let's use 3 and 4. We write these factors on the tree under the 12.



The factor 3 is prime, so we circle it. The factor 4 is composite, and it factors into $2 \cdot 2$. We write these factors under the 4. Since 2 is prime, we circle both 2s.



The prime factorization is the product of the circled primes. We generally write the prime factorization in order from least to greatest.

Equation:

$$2 \cdot 2 \cdot 3 \cdot 3$$

In cases like this, where some of the prime factors are repeated, we can write prime factorization in exponential form.

Equation:

$$2 \cdot 2 \cdot 3 \cdot 3$$

$$2^2 \cdot 3^2$$

Note that we could have started our factor tree with any factor pair of 36. We chose 12 and 3, but the same result would have been the same if we had started with 2 and 18, 4 and 9, or 6 and 6.

Note:
Find the prime factorization of a composite number using the tree method.

Find any factor pair of the given number, and use these numbers to create two branches.
If a factor is prime, that branch is complete. Circle the prime.
If a factor is not prime, write it as the product of a factor pair and continue the process.
Write the composite number as the product of all the circled primes.

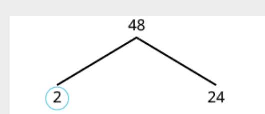
Example:
Exercise:

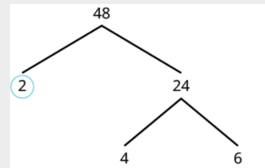
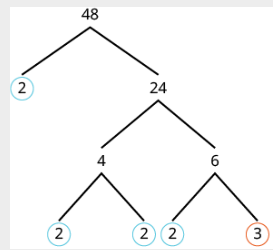
Problem:

Find the prime factorization of 48 using the factor tree method.

Solution:
Solution

We can start our tree using any factor pair of 48. Let's use 2 and 24.

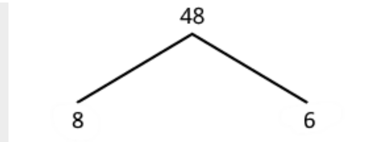
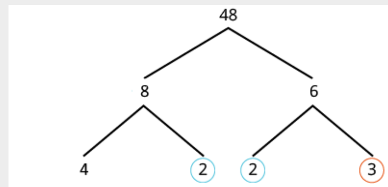
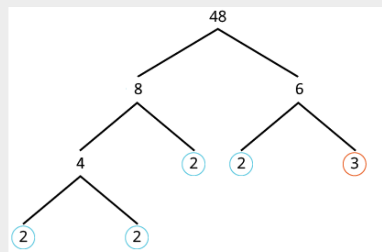


We circle the 2 because it is prime and so that branch is complete.	
Now we will factor 24. Let's use 4 and 6.	
Neither factor is prime, so we do not circle either. We factor the 4, using 2 and 2. We factor 6, using 2 and 3. We circle the 2s and the 3 since they are prime. Now all of the branches end in a prime.	
Write the product of the circled numbers.	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$
Write in exponential form.	$2^4 \cdot 3$

Check this on your own by multiplying all the factors together. The result should be 48.

Solution:
Solution

We can start our tree using any factor pair of 48. Let's use 8 and 6	

this time. Neither of these are prime so we don't circle anything.	
Now we will factor 8 using 4 and 2. We will also factor 6 using 2 and 3. We circle the 2s and 3 since they are prime. Almost done.	
We factor the 4 into 2 and 2. We circle the last pair of 2s. Now all of the branches end in a prime.	
Write the product of the circled numbers.	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$
Write in exponential form.	$2^4 \cdot 3$

The prime factors are exactly the same even though we chose different factors for the first step. It is a property of natural numbers that they always factor into the same prime factors no matter which path is followed.

Note:
Exercise:

Problem:

Find the prime factorization using the factor tree method: 80

Solution:

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$, or $2^4 \cdot 5$

Note:
Exercise:

Problem:

Find the prime factorization using the factor tree method: 60

Solution:

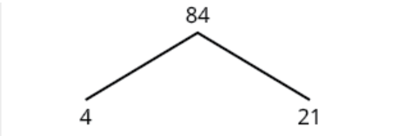
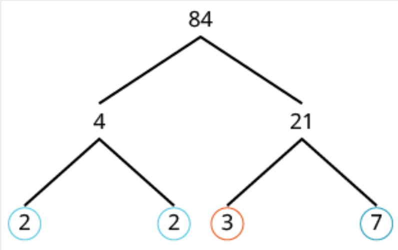
$2 \cdot 2 \cdot 3 \cdot 5$, or $2^2 \cdot 3 \cdot 5$

Example:
Exercise:

Problem:

Find the prime factorization of 84 using the factor tree method.

Solution:
Solution

We start with the factor pair 4 and 21. Neither factor is prime so we factor them further.	
Now the factors are all prime, so we circle them.	
Then we write 84 as the product of all circled primes.	$2 \cdot 2 \cdot 3 \cdot 7$ $2^2 \cdot 3 \cdot 7$

Draw a factor tree of 84.

Note:
Exercise:

Problem:

Find the prime factorization using the factor tree method: 126

Solution:

$2 \cdot 3 \cdot 3 \cdot 7$, or $2 \cdot 3^2 \cdot 7$

Note:
Exercise:

Problem: Find the prime factorization using the factor tree method: 294
Solution: $2 \cdot 3 \cdot 7 \cdot 7$, or $2 \cdot 3 \cdot 7^2$

Prime Factorization Using the Ladder Method

The ladder method is another way to find the prime factors of a composite number. It leads to the same result as the factor tree method. Some people prefer the ladder method to the factor tree method, and vice versa.

To begin building the “ladder,” divide the given number by its smallest prime factor. For example, to start the ladder for 36, we divide 36 by 2, the smallest prime factor of 36.

$$\begin{array}{r} 18 \\ 2\overline{)36} \end{array}$$

To add a “step” to the ladder, we continue dividing by the same prime until it no longer divides evenly.

$$\begin{array}{r} 9 \\ 2\overline{)18} \\ 2\overline{)\ 36} \end{array}$$

Then we divide by the next prime; so we divide 9 by 3.

$$\begin{array}{r} 3 \\ 3\overline{)9} \\ 2\overline{)18} \\ 2\overline{)\ 36} \end{array}$$

We continue dividing up the ladder in this way until the quotient is prime. Since the quotient, 3, is prime, we stop here.

Do you see why the ladder method is sometimes called stacked division?

The prime factorization is the product of all the primes on the sides and top of the ladder.
Equation:

$$2 \cdot 2 \cdot 3 \cdot 3$$

$$2^2 \cdot 3^2$$

Notice that the result is the same as we obtained with the factor tree method.

Note:
Find the prime factorization of a composite number using the ladder method.

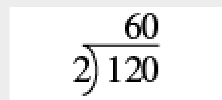
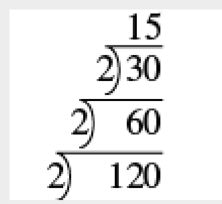
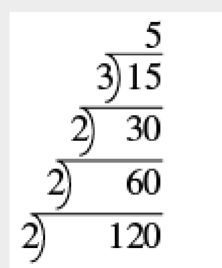
Divide the number by the smallest prime.
Continue dividing by that prime until it no longer divides evenly.
Divide by the next prime until it no longer divides evenly.
Continue until the quotient is a prime.
Write the composite number as the product of all the primes on the sides and top of the ladder.

Example:
Exercise:

Problem:

Find the prime factorization of 120 using the ladder method.

Solution:
Solution

Divide the number by the smallest prime, which is 2.	
Continue dividing by 2 until it no longer divides evenly.	
Divide by the next prime, 3.	
The quotient, 5, is prime, so the ladder is complete. Write the prime factorization of 120.	$2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$ $2^3 \cdot 3 \cdot 5$

Check this yourself by multiplying the factors. The result should be 120.

Note:
Exercise:

Problem: Find the prime factorization using the ladder method: 80

Solution:

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$, or $2^4 \cdot 5$

Note:
Exercise:

Problem: Find the prime factorization using the ladder method: 60

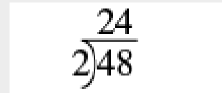
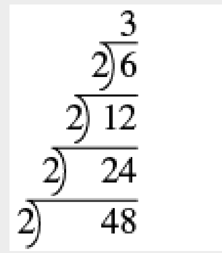
Solution:

$2 \cdot 2 \cdot 3 \cdot 5$, or $2^2 \cdot 3 \cdot 5$

Example:
Exercise:

Problem: Find the prime factorization of 48 using the ladder method.

Solution:
Solution

Divide the number by the smallest prime, 2.	
Continue dividing by 2 until it no longer divides evenly.	
The quotient, 3, is prime, so the ladder is complete. Write the prime factorization of 48.	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$ $2^4 \cdot 3$

Note:
Exercise:

Problem: Find the prime factorization using the ladder method. 126

Solution:

$2 \cdot 3 \cdot 3 \cdot 7$, or $2 \cdot 3^2 \cdot 7$

Note:
Exercise:

Problem: Find the prime factorization using the ladder method. 294

Solution:

$2 \cdot 3 \cdot 7 \cdot 7$, or $2 \cdot 3 \cdot 7^2$

Find the Least Common Multiple (LCM) of Two Numbers

One of the reasons we look at multiples and primes is to use these techniques to find the least common multiple of two numbers. This will be useful when we add and subtract fractions with different denominators.

Listing Multiples Method

A common multiple of two numbers is a number that is a multiple of both numbers. Suppose we want to find common multiples of 10 and 25. We can list the first several multiples of each number. Then we look for multiples that are common to both lists—these are the common multiples.
Equation:

$$\begin{aligned}&10{:}10, 20, 30, 40,\ 50, 60, 70, 80, 90, 100, 110, \ldots\\&25{:}25,\ 50, 75,\ 100, 125, \ldots\end{aligned}$$

We see that 50 and 100 appear in both lists. They are common multiples of 10 and 25. We would find more common multiples if we continued the list of multiples for each.

The smallest number that is a multiple of two numbers is called the **least common multiple** (LCM). So the least LCM of 10 and 25 is 50.

Note:
Find the least common multiple (LCM) of two numbers by listing multiples.

List the first several multiples of each number.
Look for multiples common to both lists. If there are no common multiples in the lists, write out additional multiples for each number.
Look for the smallest number that is common to both lists.
This number is the LCM.

Example:
Exercise:

Problem: Find the LCM of 15 and 20 by listing multiples.

Solution:
Solution

List the first several multiples of 15 and of 20. Identify the first common multiple.

$15\text{: } 15, 30, 45, \ 60, 75, 90, 105, 120$
$20\text{: } 20, 40, \ 60, 80, 100, 120, 140, 160$

The smallest number to appear on both lists is 60, so 60 is the least common multiple of 15 and 20.

Notice that 120 is on both lists, too. It is a common multiple, but it is not the least common multiple.

Note:
Exercise:

Problem:

Find the least common multiple (LCM) of the given numbers:
9 and 12

Solution:

36

Note:
Exercise:

Problem:

Find the least common multiple (LCM) of the given numbers:
18 and 24

Solution:

72

Prime Factors Method

Another way to find the least common multiple of two numbers is to use their prime factors. We'll use this method to find the LCM of 12 and 18.

We start by finding the prime factorization of each number.
Equation:

$$12 = 2 \cdot 2 \cdot 3 \qquad\qquad 18 = 2 \cdot 3 \cdot 3$$

Then we write each number as a product of primes, matching primes vertically when possible.
Equation:

$$\begin{aligned} 12 &= 2 \cdot 2 \cdot 3 \\ 18 &= 2 \cdot \phantom{2 \cdot{}} 3 \cdot 3 \end{aligned}$$

Now we bring down the primes in each column. The LCM is the product of these factors.

$$
\begin{array}{rcccccccl}
12 = & 2 & \cdot & 2 & \cdot & 3 & & & \\
18 = & 2 & \cdot & & & 3 & \cdot & 3 & \\
\hline
\text{LCM} = & 2 & \cdot & 2 & \cdot & 3 & \cdot & 3 & \\
\text{LCM} = & 2 & \cdot & 2 & \cdot & 3 & \cdot & 3 & = 36
\end{array}
$$

Notice that the prime factors of 12 and the prime factors of 18 are included in the LCM. By matching up the common primes, each common prime factor is used only once. This ensures that 36 is the least common multiple.

Note:
Find the LCM using the prime factors method.

Find the prime factorization of each number.
Write each number as a product of primes, matching primes vertically when possible.
Bring down the primes in each column.
Multiply the factors to get the LCM.

Example:
Exercise:

Problem:

Find the LCM of 15 and 18 using the prime factors method.

Solution:
Solution

Write each number as a product of primes.	$15 = 3 \cdot 5 \qquad 18 = 2 \cdot 3 \cdot 3$
Write each number as a product of primes, matching primes vertically when possible.	$\begin{array}{rcccc} 15 = & & 3\ \cdot & & 5 \\ 18 = & 2\ \cdot & 3\ \cdot & 3 & \end{array}$
Bring down the primes in each column.	$\begin{array}{rcccc} 15 = & & 3\ \cdot & & 5 \\ 18 = & 2\ \cdot & 3\ \cdot & 3 & \\ \hline \text{LCM} = & 2\ \cdot & 3\ \cdot & 3\ \cdot & 5 \end{array}$
Multiply the factors to get the LCM.	$\text{LCM} = 2 \cdot 3 \cdot 3 \cdot 5$ The LCM of 15 and 18 is 90.

Note:
Exercise:

Problem: Find the LCM using the prime factors method. 15 and 20

Solution:

60

Note:
Exercise:

Problem: Find the LCM using the prime factors method. 15 and 35

Solution:

105

Example:
Exercise:

Problem:

Find the LCM of 50 and 100 using the prime factors method.

Solution:
Solution

Write the prime factorization of each number.	$50 = 2 \cdot 5 \cdot 5 \qquad 100 = 2 \cdot 2 \cdot 5 \cdot 5$
Write each number as a product of primes, matching primes vertically when possible.	$\begin{array}{l} 50 = \phantom{2 \cdot{}} 2 \cdot 5 \cdot 5 \\ 100 = 2 \cdot 2 \cdot 5 \cdot 5 \end{array}$
Bring down the primes in each column.	$\begin{array}{l} 50 = \phantom{2 \cdot{}} 2 \cdot 5 \cdot 5 \\ 100 = 2 \cdot 2 \cdot 5 \cdot 5 \\ \hline \text{LCM} = 2 \cdot 2 \cdot 5 \cdot 5 \end{array}$
Multiply the factors to get the LCM.	$\text{LCM} = 2 \cdot 2 \cdot 5 \cdot 5$ The LCM of 50 and 100 is 100.

Note:
Exercise:

Problem: Find the LCM using the prime factors method: $55, 88$

Solution:

440

Note:
Exercise:

Problem: Find the LCM using the prime factors method: $60, 72$

Solution:

360

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Ex 1: Prime Factorization
- Ex 2: Prime Factorization
- Ex 3: Prime Factorization
- Ex 1: Prime Factorization Using Stacked Division
- Ex 2: Prime Factorization Using Stacked Division
- The Least Common Multiple
- Example: Determining the Least Common Multiple Using a List of Multiples
- Example: Determining the Least Common Multiple Using Prime Factorization

Key Concepts

- **Find the prime factorization of a composite number using the tree method.**

  Find any factor pair of the given number, and use these numbers to create two branches.
  If a factor is prime, that branch is complete. Circle the prime.
  If a factor is not prime, write it as the product of a factor pair and continue the process.
  Write the composite number as the product of all the circled primes.

- **Find the prime factorization of a composite number using the ladder method.**

  Divide the number by the smallest prime.
  Continue dividing by that prime until it no longer divides evenly.
  Divide by the next prime until it no longer divides evenly.
  Continue until the quotient is a prime.
  Write the composite number as the product of all the primes on the sides and top of the ladder.

- **Find the LCM by listing multiples.**

  List the first several multiples of each number.
  Look for multiples common to both lists. If there are no common multiples in the lists, write out additional multiples for each number.
  Look for the smallest number that is common to both lists.
  This number is the LCM.

- **Find the LCM using the prime factors method.**

  Find the prime factorization of each number.
  Write each number as a product of primes, matching primes vertically when possible.
  Bring down the primes in each column.
  Multiply the factors to get the LCM.

Section Exercises

Practice Makes Perfect

Find the Prime Factorization of a Composite Number

In the following exercises, find the prime factorization of each number using the factor tree method.

Exercise:

Problem: 86

Solution:

$2 \cdot 43$

Exercise:

Problem: 78

Exercise:

Problem: 132

Solution:

$2 \cdot 2 \cdot 3 \cdot 11$

Exercise:

Problem: 455

Exercise:

Problem: 693

Solution:

$3 \cdot 3 \cdot 7 \cdot 11$

Exercise:

Problem: 420

Exercise:

Problem: 115

Solution:

$5 \cdot 23$

Exercise:

Problem: 225

Exercise:

Problem: 2475

Solution:

$3 \cdot 3 \cdot 5 \cdot 5 \cdot 11$

Exercise:

Problem: 1560

In the following exercises, find the prime factorization of each number using the ladder method.

Exercise:

Problem: 56

Solution:

$2 \cdot 2 \cdot 2 \cdot 7$

Exercise:

Problem: 72

Exercise:

Problem: 168

Solution:

$2 \cdot 2 \cdot 2 \cdot 3 \cdot 7$

Exercise:

Problem: 252

Exercise:

Problem: 391

Solution:

$17 \cdot 23$

Exercise:

Problem: 400

Exercise:

Problem: 432

Solution:

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

Exercise:

Problem: 627

Exercise:

Problem: 2160

Solution:

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5$

Exercise:

Problem: 2520

In the following exercises, find the prime factorization of each number using any method.

Exercise:

Problem: 150

Solution:

$2 \cdot 3 \cdot 5 \cdot 5$

Exercise:

Problem: 180

Exercise:

Problem: 525

Solution:

$3 \cdot 5 \cdot 5 \cdot 7$

Exercise:

Problem: 444

Exercise:

Problem: 36

Solution:

$2 \cdot 2 \cdot 3 \cdot 3$

Exercise:

Problem: 50

Exercise:

Problem: 350

Solution:

$2 \cdot 5 \cdot 5 \cdot 7$

Exercise:

Problem: 144

Find the Least Common Multiple (LCM) of Two Numbers

In the following exercises, find the least common multiple (LCM) by listing multiples.

Exercise:

Problem: $8, 12$

Solution:

24

Exercise:

Problem: $4, 3$

Exercise:

Problem: $6, 15$

Solution:

30

Exercise:

Problem: $12, 16$

Exercise:

Problem: $30, 40$

Solution:

120

Exercise:

Problem: $20, 30$

Exercise:

Problem: $60, 75$

Solution:

300

Exercise:

Problem: $44, 55$

In the following exercises, find the least common multiple (LCM) by using the prime factors method.

Exercise:

Problem: $8, 12$

Solution:

24

Exercise:

Problem: $12, 16$

Exercise:

Problem: $24, 30$

Solution:

120

Exercise:

Problem: $28, 40$

Exercise:

Problem: $70, 84$

Solution:

420

Exercise:

Problem: $84, 90$

In the following exercises, find the least common multiple (LCM) using any method.

Exercise:

Problem: $6, 21$

Solution:

42

Exercise:

Problem: $9, 15$

Exercise:

Problem: $24, 30$

Solution:

120

Exercise:

Problem: $32, 40$

Everyday Math

Exercise:

Problem:

Grocery shopping Hot dogs are sold in packages of ten, but hot dog buns come in packs of eight. What is the smallest number of hot dogs and buns that can be purchased if you want to have the same number of hot dogs and buns? (Hint: it is the LCM!)

Solution:

40

Exercise:

Problem:

Grocery shopping Paper plates are sold in packages of 12 and party cups come in packs of 8. What is the smallest number of plates and cups you can purchase if you want to have the same number of each? (Hint: it is the LCM!)

Writing Exercises

Exercise:

Problem:

Do you prefer to find the prime factorization of a composite number by using the factor tree method or the ladder method? Why?

Exercise:

Problem:

Do you prefer to find the LCM by listing multiples or by using the prime factors method? Why?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the prime factorization of a composite number.			
find the least common multiple (LCM) of two numbers.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next Chapter? Why or why not?

Chapter Review Exercises

Use the Language of Algebra

Use Variables and Algebraic Symbols

In the following exercises, translate from algebra to English.

Exercise:

Problem: $3 \cdot 8$

Solution:

the product of 3 and 8

Exercise:

Problem: $12 - x$

Exercise:

Problem: $24 \div 6$

Solution:

the quotient of 24 and 6

Exercise:

Problem: $9 + 2a$

Exercise:

Problem: $50 \geq 47$

Solution:

50 is greater than or equal to 47

Exercise:

Problem: $3y < 15$

Exercise:

Problem: $n + 4 = 13$

Solution:

The sum of *n* and 4 is equal to 13

Exercise:

Problem: $32 - k = 7$

Identify Expressions and Equations

In the following exercises, determine if each is an expression or equation.

Exercise:

Problem: $5 + u = 84$

Solution:

equation

Exercise:

Problem: $36 - 6s$

Exercise:

Problem: $4y - 11$

Solution:

expression

Exercise:

Problem: $10x = 120$

Simplify Expressions with Exponents

In the following exercises, write in exponential form.

Exercise:

Problem: $2 \cdot 2 \cdot 2$

Solution:

2^3

Exercise:

Problem: $a \cdot a \cdot a \cdot a \cdot a$

Exercise:

Problem: $x \cdot x \cdot x \cdot x \cdot x \cdot x$

Solution:

x^6

Exercise:

Problem: $10 \cdot 10 \cdot 10$

In the following exercises, write in expanded form.

Exercise:

Problem: 8^4

Solution:

$8 \cdot 8 \cdot 8 \cdot 8$

Exercise:

Problem: 3^6

Exercise:

Problem: y^5

Solution:

$y \cdot y \cdot y \cdot y \cdot y$

Exercise:

Problem: n^4

In the following exercises, simplify each expression.
Exercise:

Problem: 3^4

Solution:

81

Exercise:

Problem: 10^6

Exercise:

Problem: 2^7

Solution:

128

Exercise:

Problem: 4^3

Simplify Expressions Using the Order of Operations

In the following exercises, simplify.
Exercise:

Problem: $10 + 2 \cdot 5$

Solution:

20

Exercise:

Problem: $(10 + 2) \cdot 5$

Exercise:

Problem: $(30 + 6) \div 2$

Solution:

18

Exercise:

Problem: $30 + 6 \div 2$

Exercise:

Problem: $7^2 + 5^2$

Solution:

74

Exercise:

Problem: $(7 + 5)^2$

Exercise:

Problem: $4 + 3(10 - 1)$

Solution:

31

Exercise:

Problem: $(4 + 3)(10 - 1)$

Evaluate, Simplify, and Translate Expressions

Evaluate an Expression

In the following exercises, evaluate the following expressions.

Exercise:

Problem: $9x - 5$ when $x = 7$

Solution:

58

Exercise:

Problem: y^3 when $y = 5$

Exercise:

Problem: $3a - 4b$ when $a = 10, b = 1$

Solution:

26

Exercise:

Problem: bh when $b = 7, h = 8$

Identify Terms, Coefficients and Like Terms

In the following exercises, identify the terms in each expression.

Exercise:

Problem: $12n^2 + 3n + 1$

Solution:

$12n^2$,$3n$, 1

Exercise:

Problem: $4x^3 + 11x + 3$

In the following exercises, identify the coefficient of each term.

Exercise:

Problem: $6y$

Solution:

6

Exercise:

Problem: $13x^2$

In the following exercises, identify the like terms.

Exercise:

Problem: $5x^2, 3, 5y^2, 3x, x, 4$

Solution:

3, 4, and $3x$, x

Exercise:

Problem: $8, 8r^2, 8r, 3r, r^2, 3s$

Simplify Expressions by Combining Like Terms

In the following exercises, simplify the following expressions by combining like terms.

Exercise:

Problem: $15a + 9a$

Solution:

$24a$

Exercise:

Problem: $12y + 3y + y$

Exercise:

Problem: $4x + 7x + 3x$

Solution:

$14x$

Exercise:

Problem: $6 + 5c + 3$

Exercise:

Problem: $8n + 2 + 4n + 9$

Solution:

$12n + 11$

Exercise:

Problem: $19p + 5 + 4p - 1 + 3p$

Exercise:

Problem: $7y^2 + 2y + 11 + 3y^2 - 8$

Solution:

$10y^2 + 2y + 3$

Exercise:

Problem: $13x^2 - x + 6 + 5x^2 + 9x$

Translate English Phrases to Algebraic Expressions

In the following exercises, translate the following phrases into algebraic expressions.

Exercise:

Problem: the difference of x and 6

Solution:

$x - 6$

Exercise:

Problem: the sum of 10 and twice a

Exercise:

Problem: the product of $3n$ and 9

Solution:

$3n \cdot 9$

Exercise:

Problem: the quotient of s and 4

Exercise:

Problem: 5 times the sum of y and 1

Solution:

$5(y + 1)$

Exercise:

Problem: 10 less than the product of 5 and z

Exercise:

Problem:

Jack bought a sandwich and a coffee. The cost of the sandwich was $\$3$ more than the cost of the coffee. Call the cost of the coffee c. Write an expression for the cost of the sandwich.

Solution:

$c + 3$

Exercise:

Problem:

The number of poetry books on Brianna's bookshelf is 5 less than twice the number of novels. Call the number of novels n. Write an expression for the number of poetry books.

Solve Equations Using the Subtraction and Addition Properties of Equality

Determine Whether a Number is a Solution of an Equation

In the following exercises, determine whether each number is a solution to the equation.

Exercise:

Problem: $y+16=40$

ⓐ24
ⓑ56

Solution:

ⓐ yes
ⓑ no

Exercise:

Problem: $d-6=21$

ⓐ15
ⓑ27

Exercise:

Problem: $4n+12=36$

ⓐ6
ⓑ12

Solution:

ⓐ yes
ⓑ no

Exercise:

Problem: $20q-10=70$

ⓐ3
ⓑ4

Exercise:

Problem: $15x - 5 = 10x + 45$

ⓐ2
ⓑ10

Solution:

ⓐ no
ⓑ yes

Exercise:

Problem: $22p - 6 = 18p + 86$

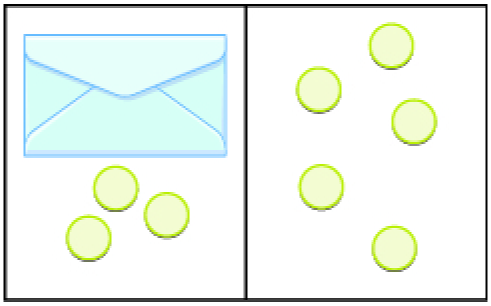
ⓐ4
ⓑ23

Model the Subtraction Property of Equality

In the following exercises, write the equation modeled by the envelopes and counters and then solve the equation using the subtraction property of equality.

Exercise:

Problem:

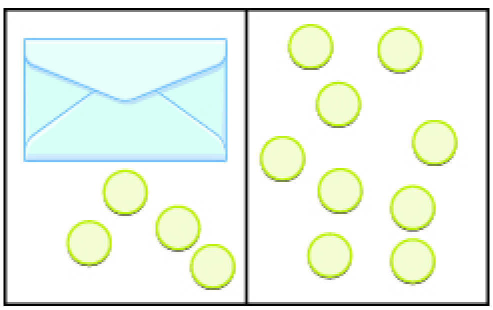


Solution:

$x + 3 = 5$; $x = 2$

Exercise:

Problem:



Solve Equations using the Subtraction Property of Equality

In the following exercises, solve each equation using the subtraction property of equality.

Exercise:

Problem: $c + 8 = 14$

Solution:

6

Exercise:

Problem: $v + 8 = 150$

Exercise:

Problem: $23 = x + 12$

Solution:

11

Exercise:

Problem: $376 = n + 265$

Solve Equations using the Addition Property of Equality

In the following exercises, solve each equation using the addition property of equality.

Exercise:

Problem: $y - 7 = 16$

Solution:

23

Exercise:

Problem: $k - 42 = 113$

Exercise:

Problem: $19 = p - 15$

Solution:

34

Exercise:

Problem: $501 = u - 399$

Translate English Sentences to Algebraic Equations

In the following exercises, translate each English sentence into an algebraic equation.

Exercise:

Problem: The sum of 7 and 33 is equal to 40.

Solution:

7 + 33 = 44

Exercise:

Problem: The difference of 15 and 3 is equal to 12.

Exercise:

Problem: The product of 4 and 8 is equal to 32.

Solution:

4 · 8 = 32

Exercise:

Problem: The quotient of 63 and 9 is equal to 7.

Exercise:

Problem: Twice the difference of n and 3 gives 76.

Solution:

$2(n-3)=76$

Exercise:

Problem: The sum of five times y and 4 is 89.

Translate to an Equation and Solve

In the following exercises, translate each English sentence into an algebraic equation and then solve it.

Exercise:

Problem: Eight more than x is equal to 35.

Solution:

$x+8=35;\ x=27$

Exercise:

Problem: 21 less than a is 11.

Exercise:

Problem: The difference of q and 18 is 57.

Solution:

$q-18=57;\ q=75$

Exercise:

Problem: The sum of m and 125 is 240.

Mixed Practice

In the following exercises, solve each equation.

Exercise:

Problem: $h - 15 = 27$

Solution:

$h = 42$

Exercise:

Problem: $k - 11 = 34$

Exercise:

Problem: $z + 52 = 85$

Solution:

$z = 33$

Exercise:

Problem: $x + 93 = 114$

Exercise:

Problem: $27 = q + 19$

Solution:

$q = 8$

Exercise:

Problem: $38 = p + 19$

Exercise:

Problem: $31 = v - 25$

Solution:

$v = 56$

Exercise:

Problem: $38 = u - 16$

Find Multiples and Factors

Identify Multiples of Numbers

In the following exercises, list all the multiples less than 50 for each of the following.

Exercise:

Problem: 3

Solution:

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48

Exercise:

Problem: 2

Exercise:

Problem: 8

Solution:

8, 16, 24, 32, 40, 48

Exercise:

Problem: 10

Use Common Divisibility Tests

In the following exercises, using the divisibility tests, determine whether each number is divisible by 2, by 3, by 5, by 6, and by 10.

Exercise:

Problem: 96

Solution:

2, 3, 6

Exercise:

Problem: 250

Exercise:

Problem: 420

Solution:

2, 3, 5, 6, 10

Exercise:

Problem: 625

Find All the Factors of a Number

In the following exercises, find all the factors of each number.

Exercise:

Problem: 30

Solution:

1, 2, 3, 5, 6, 10, 15, 30

Exercise:

Problem: 70

Exercise:

Problem: 180

Solution:

1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, 180

Exercise:

Problem: 378

Identify Prime and Composite Numbers

In the following exercises, identify each number as prime or composite.

Exercise:

Problem: 19

Solution:

prime

Exercise:

Problem: 51

Exercise:

Problem: 121

Solution:

composite

Exercise:

Problem: 219

Prime Factorization and the Least Common Multiple

Find the Prime Factorization of a Composite Number

In the following exercises, find the prime factorization of each number.

Exercise:

Problem: 84

Solution:

$2 \cdot 2 \cdot 3 \cdot 7$

Exercise:

Problem: 165

Exercise:

Problem: 350

Solution:

$2 \cdot 5 \cdot 5 \cdot 7$

Exercise:

Problem: 572

Find the Least Common Multiple of Two Numbers

In the following exercises, find the least common multiple of each pair of numbers.

Exercise:

Problem: $9, 15$

Solution:

45

Exercise:

Problem: $12, 20$

Exercise:

Problem: $25, 35$

Solution:

175

Exercise:

Problem: $18, 40$

Everyday Math

Exercise:

Problem:

Describe how you have used two topics from The Language of Algebra chapter in your life outside of your math class during the past month.

Solution:

Answers will vary

Chapter Practice Test

In the following exercises, translate from an algebraic equation to English phrases.

Exercise:

Problem: $6 \cdot 4$

Exercise:

Problem: $15 - x$

Solution:

fifteen minus x

In the following exercises, identify each as an expression or equation.

Exercise:

Problem: $5 \cdot 8 + 10$

Exercise:

Problem: $x + 6 = 9$

Solution:

equation

Exercise:

Problem: $3 \cdot 11 = 33$

Exercise:

Problem:

ⓐ Write $n \cdot n \cdot n \cdot n \cdot n \cdot n$ in exponential form.
ⓑ Write 3^5 in expanded form and then simplify.

Solution:

ⓐ n^6
ⓑ $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$

In the following exercises, simplify, using the order of operations.

Exercise:

Problem: $4 + 3 \cdot 5$

Exercise:

Problem: $(8 + 1) \cdot 4$

Solution:

36

Exercise:

Problem: $1 + 6(3 - 1)$

Exercise:

Problem: $(8+4) \div 3+1$

Solution:

5

Exercise:

Problem: $(1+4)^2$

Exercise:

Problem: $5[2+7(9-8)]$

Solution:

45

In the following exercises, evaluate each expression.

Exercise:

Problem: $8x-3$ when $x=4$

Exercise:

Problem: y^3 when $y=5$

Solution:

125

Exercise:

Problem: $6a-2b$ when $a=5, b=7$

Exercise:

Problem: hw when $h = 12, w = 3$

Solution:

36

Exercise:

Problem: Simplify by combining like terms.

ⓐ$6x + 8x$
ⓑ$9m + 10 + m + 3$

In the following exercises, translate each phrase into an algebraic expression.

Exercise:

Problem: 5 more than x

Solution:

$x + 5$

Exercise:

Problem: the quotient of 12 and y

Exercise:

Problem: three times the difference of a and b

Solution:

$3(a - b)$

Exercise:

Problem:

Caroline has 3 fewer earrings on her left ear than on her right ear. Call the number of earrings on her right ear, r. Write an expression for the number of earrings on her left ear.

In the following exercises, solve each equation.
Exercise:

Problem: $n - 6 = 25$

Solution:

$n = 31$

Exercise:

Problem: $x + 58 = 71$

In the following exercises, translate each English sentence into an algebraic equation and then solve it.
Exercise:

Problem: 15 less than y is 32.

Solution:

$y - 15 = 32$; $y = 47$

Exercise:

Problem: the sum of a and 129 is 164.

Exercise:

Problem: List all the multiples of 4, that are less than 50.

Solution:

4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48

Exercise:

Problem: Find all the factors of 90.

Exercise:

Problem: Find the prime factorization of 1080.

Solution:

$2^3 \cdot 3^3 \cdot 5$

Exercise:

Problem: Find the LCM (Least Common Multiple) of 24 and 40.

Glossary

least common multiple
: The smallest number that is a multiple of two numbers is called the least common multiple (LCM).

prime factorization
: The prime factorization of a number is the product of prime numbers that equals the number.

Introduction to Integers
class="introduction"

The peak of Mount Everest. (credit: Gunther Hagleitner, Flickr)

At over 29,000 feet, Mount Everest stands as the tallest peak on land. Located along the border of Nepal and China, Mount Everest is also known for its extreme climate. Near the summit, temperatures never rise above freezing. Every year, climbers from around the world brave the extreme conditions in an effort to scale the tremendous height. Only some are successful. Describing the drastic change in elevation the climbers experience and the change in temperatures requires using numbers that extend both above and below zero. In this chapter, we will describe these kinds of numbers and operations using them.

Introduction to Integers
By the end of this section, you will be able to:

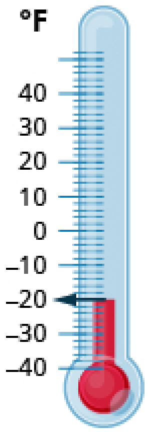
- Locate positive and negative numbers on the number line
- Order positive and negative numbers
- Find opposites
- Simplify expressions with absolute value
- Translate word phrases to expressions with integers

Note:
Before you get started, take this readiness quiz.

1. Plot $0, 1,$ and 3 on a number line.
   If you missed this problem, review [link].
2. Fill in the appropriate symbol: $(=, <,$ or $>)$: 2 ___ 4
   If you missed this problem, review [link].

Locate Positive and Negative Numbers on the Number Line

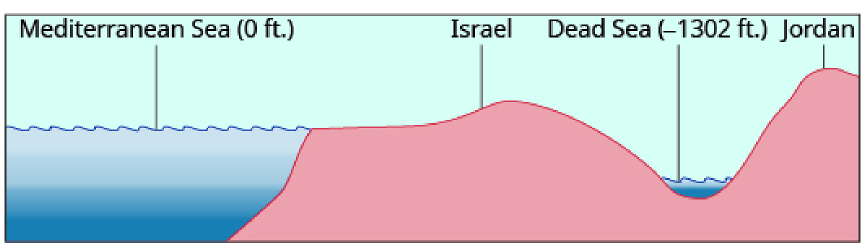
Do you live in a place that has very cold winters? Have you ever experienced a temperature below zero? If so, you are already familiar with negative numbers. A **negative number** is a number that is less than 0. Very cold temperatures are measured in degrees below zero and can be described by negative numbers. For example, -1°F (read as "negative one degree Fahrenheit") is 1 degree below 0. A minus sign is shown before a number to indicate that it is negative. [link] shows -20°F, which is 20 degrees below 0.



Temperatures below zero are described by negative numbers.

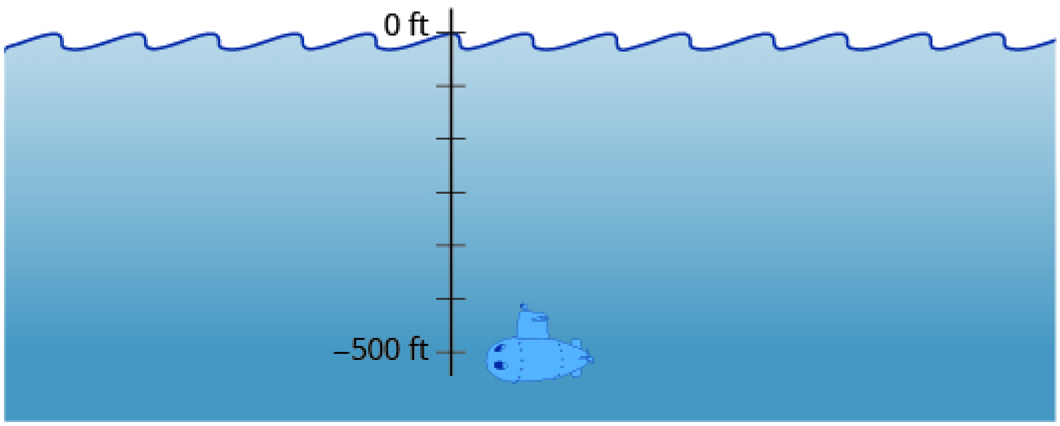
Temperatures are not the only negative numbers. A bank overdraft is another example of a negative number. If a person writes a check for more than he has in his account, his balance will be negative.

Elevations can also be represented by negative numbers. The elevation at sea level is 0 feet. Elevations above sea level are positive and elevations below sea level are negative. The elevation of the Dead Sea, which borders Israel and Jordan, is about $1{,}302$ feet below sea level, so the elevation of the Dead Sea can be represented as $-1{,}302$ feet. See [link].



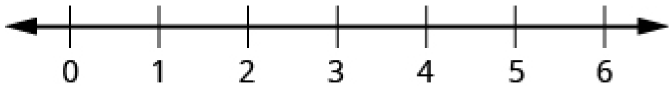
The surface of the Mediterranean Sea has an elevation of 0 ft. The diagram shows that nearby mountains have higher (positive) elevations whereas the Dead Sea has a lower (negative) elevation.

Depths below the ocean surface are also described by negative numbers. A submarine, for example, might descend to a depth of 500 feet. Its position would then be -500 feet as labeled in [link].

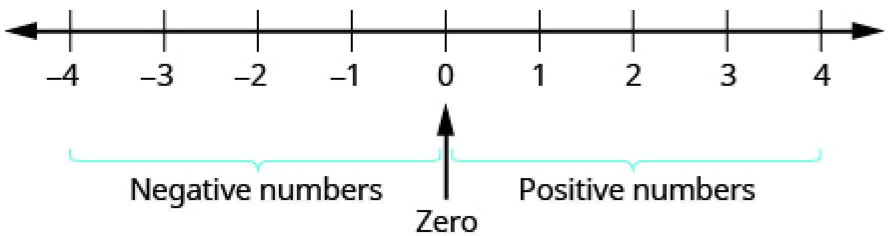


Depths below sea level are described by negative numbers. A submarine 500 ft below sea level is at -500 ft.

Both positive and negative numbers can be represented on a number line. Recall that the number line created in Add Whole Numbers started at 0 and showed the counting numbers increasing to the right as shown in [link]. The counting numbers $(1, 2, 3, \ldots)$ on the number line are all positive. We could write a plus sign, $+$, before a positive number such as $+2$ or $+3$, but it is customary to omit the plus sign and write only the number. If there is no sign, the number is assumed to be positive.



Now we need to extend the number line to include negative numbers. We mark several units to the left of zero, keeping the intervals the same width as those on the positive side. We label the marks with negative numbers, starting with -1 at the first mark to the left of 0, -2 at the next mark, and so on. See [link].



On a number line, positive numbers are to the right of zero. Negative numbers are to the left of zero. What about zero? Zero is neither positive nor negative.

The arrows at either end of the line indicate that the number line extends forever in each direction. There is no greatest positive number and there is no smallest negative number.

Example:
Exercise:

Problem: Plot the numbers on a number line:

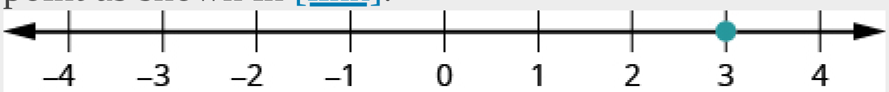
ⓐ 3
ⓑ -3

ⓒ -2

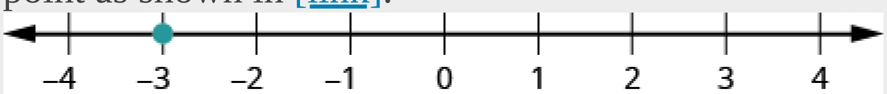
Solution:
Solution

Draw a number line. Mark 0 in the center and label several units to the left and right.

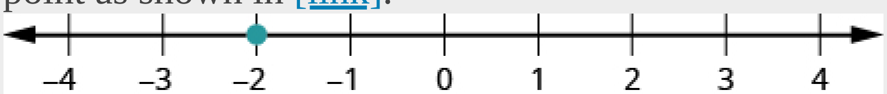
ⓐ To plot 3, start at 0 and count three units to the right. Place a point as shown in [link].



ⓑ To plot -3, start at 0 and count three units to the left. Place a point as shown in [link].



ⓒ To plot -2, start at 0 and count two units to the left. Place a point as shown in [link].

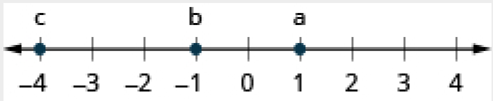


Note:
Exercise:

Problem: Plot the numbers on a number line.

ⓐ 1
ⓑ -1
ⓒ -4

Solution:

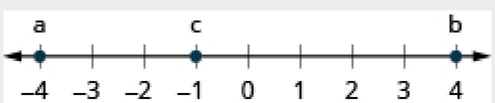


Note:
Exercise:

Problem: Plot the numbers on a number line.

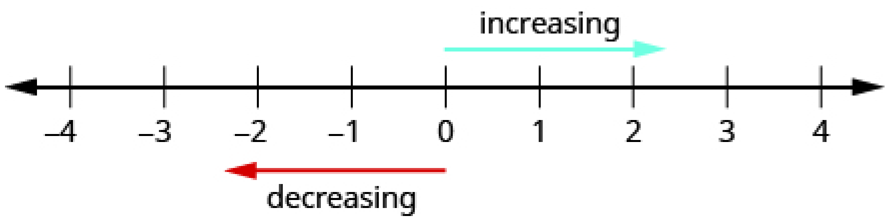
ⓐ -4
ⓑ 4
ⓐ -1

Solution:

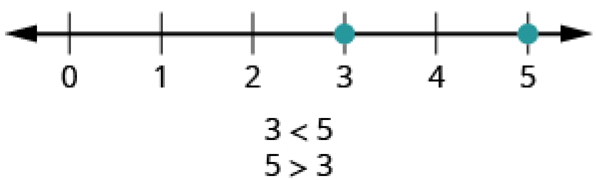


Order Positive and Negative Numbers

We can use the number line to compare and order positive and negative numbers. Going from left to right, numbers increase in value. Going from right to left, numbers decrease in value. See [link].



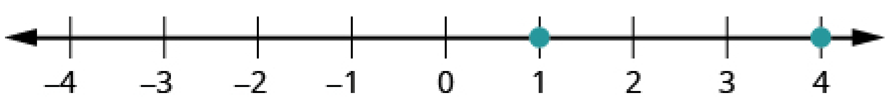
Just as we did with positive numbers, we can use inequality symbols to show the ordering of positive and negative numbers. Remember that we use the notation $a < b$ (read a *is less than* b) when a is to the left of b on the number line. We write $a > b$ (read a *is greater than* b) when a is to the right of b on the number line. This is shown for the numbers 3 and 5 in [link].



The number 3 is to the left of 5 on the number line. So 3 is less than 5, and 5 is greater than 3.

The numbers lines to follow show a few more examples.

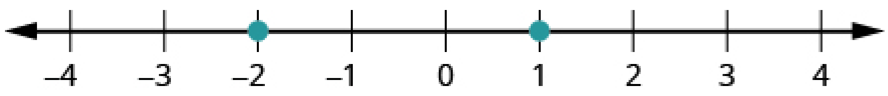
ⓐ



4 is to the right of 1 on the number line, so $4 > 1$.

1 is to the left of 4 on the number line, so $1 < 4$.

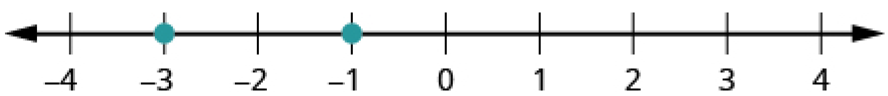
ⓑ



-2 is to the left of 1 on the number line, so $-2 < 1$.

1 is to the right of -2 on the number line, so $1 > -2$.

ⓒ



-1 is to the right of -3 on the number line, so $-1 > -3$.

-3 is to the left of -1 on the number line, so $-3 < -1$.

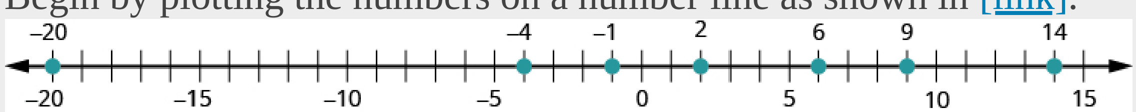
Example:
Exercise:

Problem: Order each of the following pairs of numbers using $<$ or $>$:

ⓐ 14 ____ 6
ⓑ -1 ____ 9
ⓒ -1 ____ -4
ⓓ 2 ____ -20

Solution:
Solution

Begin by plotting the numbers on a number line as shown in [link].



ⓐ Compare 14 and 6.	$14 \_\_\_ 6$
14 is to the right of 6 on the number line.	$14 > 6$

ⓑ Compare −1 and 9.	$-1 \_\_\_ 9$
−1 is to the left of 9 on the number line.	$-1 < 9$

ⓒ Compare −1 and −4.	$-1 \_\_\_ -4$
−1 is to the right of −4 on the number line.	$-1 > -4$

ⓓ Compare 2 and −20.	$-2 \_\_\_ -20$
2 is to the right of −20 on the number line.	$2 > -20$

Note:

Exercise:

Problem: Order each of the following pairs of numbers using $<$ or $>$.

ⓐ 15 ___ 7
ⓑ -2 ___ 5
ⓒ -3 ___ -7
ⓓ 5 ___ -17

Solution:

ⓐ $>$
ⓑ $<$
ⓒ $>$
ⓓ $>$

Note:
Exercise:

Problem: Order each of the following pairs of numbers using $<$ or $>$.

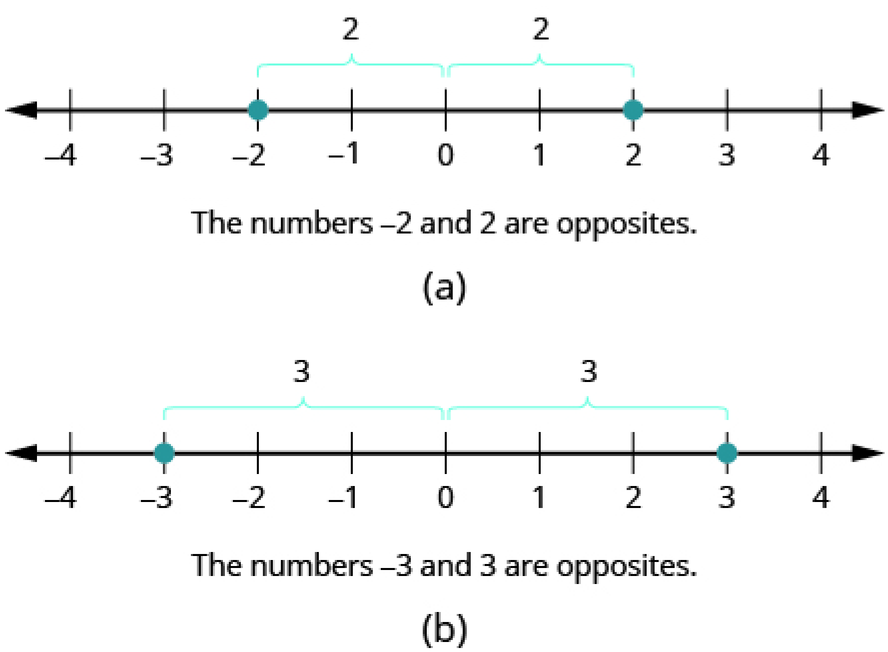
ⓐ 8 ___ 13
ⓑ 3 ___ -4
ⓒ -5 ___ -2
ⓓ 9 ___ -21

Solution:

ⓐ $<$
ⓑ $>$
ⓒ $<$
ⓓ $>$

Find Opposites

On the number line, the negative numbers are a mirror image of the positive numbers with zero in the middle. Because the numbers 2 and -2 are the same distance from zero, they are called **opposites**. The opposite of 2 is -2, and the opposite of -2 is 2 as shown in [link](a). Similarly, 3 and -3 are opposites as shown in [link](b).



Note:
Opposite
The opposite of a number is the number that is the same distance from zero on the number line, but on the opposite side of zero.

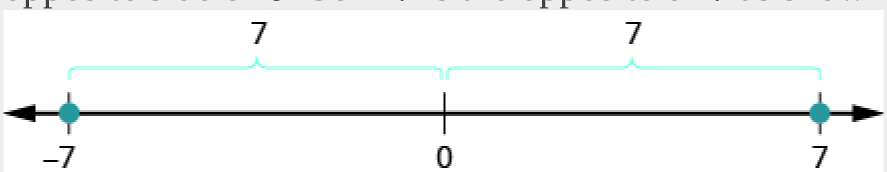
Example:
Exercise:

Problem: Find the opposite of each number:

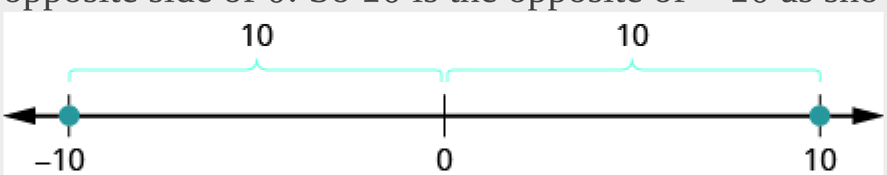
ⓐ 7
ⓑ -10

Solution:
Solution

ⓐ The number -7 is the same distance from 0 as 7, but on the opposite side of 0. So -7 is the opposite of 7 as shown in [link].



ⓑ The number 10 is the same distance from 0 as -10, but on the opposite side of 0. So 10 is the opposite of -10 as shown in [link].



Note:
Exercise:

Problem: Find the opposite of each number:

ⓐ 4
ⓑ -3

Solution:

ⓐ -4
ⓑ 3

Note:
Exercise:

Problem: Find the opposite of each number:

ⓐ 8
ⓑ -5

Solution:

ⓐ -8
ⓑ 5

Opposite Notation

Just as the same word in English can have different meanings, the same symbol in algebra can have different meanings. The specific meaning becomes clear by looking at how it is used. You have seen the symbol “$-$”, in three different ways.

$10-4$	Between two numbers, the symbol indicates the operation of subtraction. We read $10-4$ as 10 *minus* 4.
-8	In front of a number, the symbol indicates a negative number. We read -8 as *negative eight*.
$-x$	In front of a variable or a number, it indicates the opposite. We read$-x$ as *the opposite of* x.
$-(-2)$	Here we have two signs. The sign in the parentheses indicates that the number is negative 2. The sign outside the parentheses indicates the opposite. We read $-(-2)$ as *the opposite of* -2.

Note:
Opposite Notation
$-a$ means the opposite of the number a
The notation $-a$ is read *the opposite of* a.

Example:
Exercise:

Problem: Simplify: $-(-6)$.

Solution:
Solution

	$-(-6)$
The opposite of -6 is 6.	6

Note:
Exercise:

Problem: Simplify:

$-(-1)$

Solution:

1

Note:
Exercise:

Problem: Simplify:

$-(-5)$

Solution:

5

Integers

The set of counting numbers, their opposites, and 0 is the set of integers.

Note:
Integers
Integers are counting numbers, their opposites, and zero.
Equation:

$$\ldots -3, -2, -1, 0, 1, 2, 3\ldots$$

We must be very careful with the signs when evaluating the opposite of a variable.

Example:
Exercise:

Problem: Evaluate $-x$:

ⓐ when $x = 8$
ⓑ when $x = -8$.

Solution:

ⓐ To evaluate $-x$ when $x = 8$, substitute 8 for x.	
	$-x$

Substitute 8 for x.	$-(8)$
Simplify.	-8

ⓑ To evaluate $-x$ when $x=-8$, substitute -8 for x.	
	$-x$
Substitute −8 for x.	$-(-8)$
Simplify.	8

Note:
Exercise:

Problem: Evaluate $-n$:

ⓐ when $n=4$
ⓑ when $n=-4$

Solution:

ⓐ −4

ⓑ 4

Note:
Exercise:

Problem: Evaluate: $-m$:

ⓐ when $m = 11$
ⓑ when $m = -11$

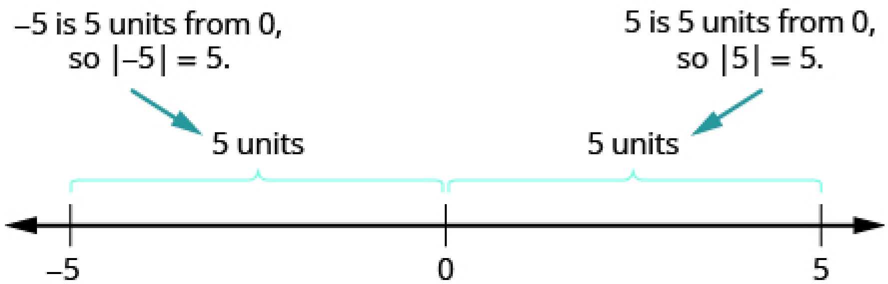
Solution:

ⓐ −11
ⓑ 11

Simplify Expressions with Absolute Value

We saw that numbers such as 5 and -5 are opposites because they are the same distance from 0 on the number line. They are both five units from 0. The distance between 0 and any number on the number line is called the **absolute value** of that number. Because distance is never negative, the absolute value of any number is never negative.

The symbol for absolute value is two vertical lines on either side of a number. So the absolute value of 5 is written as $|5|$, and the absolute value of -5 is written as $|-5|$ as shown in [link].



Note:
Absolute Value
The absolute value of a number is its distance from 0 on the number line. The absolute value of a number n is written as $|n|$.
Equation:

$$|n| \geq 0 \text{ for all numbers}$$

Example:
Exercise:

Problem: Simplify:

ⓐ $|3|$
ⓑ $|-44|$
ⓒ $|0|$

Solution:
Solution

ⓐ	
	$\lvert 3\rvert$
3 is 3 units from zero.	$\mathbf{3}$

ⓑ	
	$\lvert -44\rvert$
−44 is 44 units from zero.	44

ⓒ	
	$\lvert 0\rvert$
0 is already at zero.	0

Note:
Exercise:

Problem: Simplify:

ⓐ $|12|$
ⓑ $-|-28|$

Solution:

ⓐ 12
ⓑ −28

Note:
Exercise:

Problem: Simplify:

ⓐ $|9|$
ⓑ (b) $-|37|$

Solution:

ⓐ 9
ⓑ −37

We treat absolute value bars just like we treat parentheses in the order of operations. We simplify the expression inside first.

Example:
Exercise:

Problem: Evaluate:

ⓐ $|x|$ when $x = -35$
ⓑ $|-y|$ when $y = -20$
ⓒ $-|u|$ when $u = 12$
ⓓ $-|p|$ when $p = -14$

Solution:
Solution

ⓐ To find $|x|$ when $x = -35$:	
	$|x|$
Substitute -35 for x.	$|-35|$
Take the absolute value.	35

ⓑ To find $|-y|$ when $y = -20$:	
	$|-y|$

Substitute -20 for y.	$\|-(-20)\|$
Simplify.	$\|20\|$
Take the absolute value.	20

ⓒ To find $-\|u\|$ when $u = 12$:	
	$-\|u\|$
Substitute 12 for u.	$-\|12\|$
Take the absolute value.	-12

ⓓ To find $-\|p\|$ when $p = -14$:	
	$-\|p\|$
Substitute -14 for p.	$-\|-14\|$
Take the absolute value.	-14

Notice that the result is negative only when there is a negative sign outside the absolute value symbol.

Note:
Exercise:

Problem: Evaluate:

ⓐ $|x|$ when $x = -17$
ⓑ $|-y|$ when $y = -39$
ⓒ $-|m|$ when $m = 22$
ⓓ $-|p|$ when $p = -11$

Solution:

ⓐ 17
ⓑ 39
ⓒ −22
ⓓ −11

Note:
Exercise:

Problem:

ⓐ $|y|$ when $y = -23$
ⓑ $|-y|$ when $y = -21$
ⓒ $-|n|$ when $n = 37$
ⓓ $-|q|$ when $q = -49$

Solution:

ⓐ 23
ⓑ 21
ⓒ −37
ⓓ −49

Example:
Exercise:

Problem: Fill in $<, >,$ or $=$ for each of the following:

ⓐ $|-5|$ ___ $-|-5|$
ⓑ 8 ___ $-|-8|$
ⓒ -9 ___ $-|-9|$
ⓓ $-|-7|$ ___ -7

Solution:
Solution

To compare two expressions, simplify each one first. Then compare.

ⓐ	
	$\|-5\|$ ___ $-\|-5\|$
Simplify.	5 ___ -5

Order.	$5 > -5$

ⓑ	
	$8 \_\_\_ -\|-8\|$
Simplify.	$8 \_\_\_ -8$
Order.	$8 > -8$

ⓒ	
	$-9 \_\_\_ -\|-9\|$
Simplify.	$-9 \_\_\_ -9$
Order.	$-9 = -9$

ⓓ	
	$-\|-7\| \_\_\_ -7$

Simplify.	-7 ___ -7
Order.	$-7 = -7$

Note:
Exercise:

Problem: Fill in $<, >,$ or $=$ for each of the following:

ⓐ $|-9|$ ___ $-|-9|$
ⓑ 2 ___ $-|-2|$
ⓒ -8 ___ $|-8|$
ⓓ $-|-5|$___-5

Solution:

ⓐ $>$
ⓑ $>$
ⓒ $<$
ⓓ $=$

Note:
Exercise:

Problem: Fill in $<, >,$ or $=$ for each of the following:

ⓐ 7___ $-|-7|$
ⓑ $-|-11|$___-11

ⓒ $|-4|\_\_\_\_ -|-4|$
ⓓ $-1\_\_\_\_|-1|$

Solution:

ⓐ$>$
ⓑ $=$
ⓒ $>$
ⓓ $<$

Absolute value bars act like grouping symbols. First simplify inside the absolute value bars as much as possible. Then take the absolute value of the resulting number, and continue with any operations outside the absolute value symbols.

Example:
Exercise:

Problem: Simplify:

ⓐ $|9-3|$
ⓑ $4|-2|$

Solution:
Solution

For each expression, follow the order of operations. Begin inside the absolute value symbols just as with parentheses.

ⓐ	
	$\lvert 9-3 \rvert$
Simplify inside the absolute value sign.	$\lvert 6 \rvert$
Take the absolute value.	6

ⓑ	
	$4\lvert -2 \rvert$
Take the absolute value.	$4 \cdot 2$
Multiply.	8

Note:
Exercise:

Problem: Simplify:

ⓐ $\lvert 12-9 \rvert$
ⓑ $3\lvert -6 \rvert$

Solution:

ⓐ 3
ⓑ 18

Note:
Exercise:

Problem: Simplify:

ⓐ $|27-16|$
ⓑ $9|-7|$

Solution:

ⓐ 11
ⓑ 63

Example:
Exercise:

Problem: Simplify: $|8+7|-|5+6|$.

Solution:
Solution

For each expression, follow the order of operations. Begin inside the absolute value symbols just as with parentheses.

	$\lvert 8+7\rvert-\lvert 5+6\rvert$
Simplify inside each absolute value sign.	$\lvert 15\rvert-\lvert 11\rvert$
Subtract.	4

Note:
Exercise:

Problem: Simplify: $\lvert 1+8\rvert - \lvert 2+5\rvert$

Solution:

2

Note:
Exercise:

Problem: Simplify: $\lvert 9-5\rvert - \lvert 7-6\rvert$

Solution:

3

Example:
Exercise:

Problem: Simplify: $24 - |19 - 3(6 - 2)|$.

Solution:
Solution

We use the order of operations. Remember to simplify grouping symbols first, so parentheses inside absolute value symbols would be first.

	$24 - \lvert 19 - 3(6 - 2)\rvert$
Simplify in the parentheses first.	$24 - \lvert 19 - 3(4)\rvert$
Multiply $3(4)$.	$24 - \lvert 19 - 12\rvert$
Subtract inside the absolute value sign.	$24 - \lvert 7\rvert$
Take the absolute value.	$24 - 7$
Subtract.	17

Note:
Exercise:

Problem: Simplify: $19 - |11 - 4(3 - 1)|$

Solution:

16

Note:
Exercise:

Problem: Simplify: $9 - |8 - 4(7 - 5)|$

Solution:

9

Translate Word Phrases into Expressions with Integers

Now we can translate word phrases into expressions with integers. Look for words that indicate a negative sign. For example, the word *negative* in “negative twenty” indicates -20. So does the word *opposite* in “the opposite of 20.”

Example:
Exercise:

Problem: Translate each phrase into an expression with integers:

ⓐ the opposite of positive fourteen
ⓑ the opposite of -11
ⓒ negative sixteen
ⓓ two minus negative seven

Solution:
Solution

ⓐ the opposite of fourteen
-14
ⓑ the opposite of −11
$-(-11)$
ⓒ negative sixteen
-16
ⓓ two minus negative seven
$2-(-7)$

Note:
Exercise:

Problem: Translate each phrase into an expression with integers:

ⓐ the opposite of positive nine
ⓑ the opposite of -15
ⓒ negative twenty
ⓓ eleven minus negative four

Solution:

ⓐ −9
ⓑ 15
ⓒ −20
ⓓ 11−(−4)

Note:
Exercise:

Problem: Translate each phrase into an expression with integers:

ⓐ the opposite of negative nineteen
ⓑ the opposite of twenty-two
ⓒ negative nine
ⓓ negative eight minus negative five

Solution:

ⓐ 19
ⓑ -22
ⓒ -9
ⓓ $-8-(-5)$

As we saw at the start of this section, negative numbers are needed to describe many real-world situations. We'll look at some more applications of negative numbers in the next example.

Example:
Exercise:

Problem: Translate into an expression with integers:

ⓐ The temperature is 12 degrees Fahrenheit below zero.
ⓑ The football team had a gain of 3 yards.
ⓒ The elevation of the Dead Sea is $1,302$ feet below sea level.
ⓓ A checking account is overdrawn by $\$40$.

Solution:
Solution

Look for key phrases in each sentence. Then look for words that indicate negative signs. Don't forget to include units of measurement described in the sentence.

ⓐ	The temperature is 12 degrees Fahrenheit below zero.
Below zero tells us that 12 is a negative number.	-12^{o}F

ⓑ	The football team had a gain of 3 yards.
A *gain* tells us that 3 is a positive number.	3 yards

ⓒ	The elevation of the Dead Sea is 1,302 feet below sea level.

Below sea level tells us that 1,302 is a negative number.	$-1,302$ feet

ⓓ	A checking account is overdrawn by \$40.
Overdrawn tells us that 40 is a negative number.	$-\$40$

Note:
Exercise:

Problem: Translate into an expression with integers:

The football team had a gain of 5 yards.

Solution:

5 yards

Note:
Exercise:

Problem: Translate into an expression with integers:

The scuba diver was 30 feet below the surface of the water.

Solution:

−30 feet

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Introduction to Integers
- Simplifying the Opposites of Negative Integers
- Comparing Absolute Value of Integers
- Comparing Integers Using Inequalities

Key Concepts

- Opposite Notation
  - $-a$ means the opposite of the number a
  - The notation $-a$ is read *the opposite of* a.
- Absolute Value Notation
  - The absolute value of a number n is written as $|n|$.
  - $|n| \geq 0$ for all numbers.

Practice Makes Perfect

Locate Positive and Negative Numbers on the Number Line

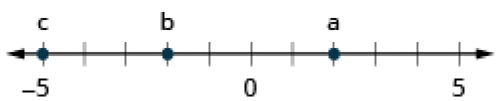
In the following exercises, locate and label the given points on a number line.

Exercise:

Problem:

ⓐ 2
ⓑ -2
ⓒ -5

Solution:



Exercise:

Problem:

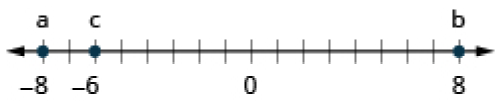
ⓐ 5
ⓑ -5
ⓒ -2

Exercise:

Problem:

ⓐ -8
ⓑ 8
ⓒ -6

Solution:



Exercise:

Problem:

ⓐ -7
ⓑ 7
ⓒ -1

Order Positive and Negative Numbers on the Number Line

In the following exercises, order each of the following pairs of numbers, using $<$ or $>$.

Exercise:

Problem:

ⓐ $9\_\_4$
ⓑ $-3\_\_6$
ⓒ $-8\_\_-2$
ⓓ $1\_\_-10$

Solution:

ⓐ $>$
ⓑ $<$
ⓒ $<$
ⓓ $>$

Exercise:

Problem:

ⓐ $6\_\_2;$
ⓑ $-7\_\_4;$

ⓒ -9 __ -1;
ⓓ 9 __ -3

Exercise:

Problem:

ⓐ -5 __ 1;
ⓑ -4 __ -9;
ⓒ 6 __ 10;
ⓓ 3 __ -8

Solution:

ⓐ $<$
ⓑ $>$
ⓒ $<$
ⓓ $>$

Exercise:

Problem:

ⓐ -7 __ 3;
ⓑ -10 __ -5;
ⓒ 2 __ -6;
ⓓ 8 __ 9

Find Opposites

In the following exercises, find the opposite of each number.

Exercise:

Problem:

ⓐ 2
ⓑ -6

Solution:

ⓐ -2
ⓑ 6

Exercise:

Problem:

ⓐ 9
ⓑ -4

Exercise:

Problem:

ⓐ -8
ⓑ 1

Solution:

ⓐ 8
ⓑ -1

Exercise:

Problem:

ⓐ -2
ⓑ 6

In the following exercises, simplify.
Exercise:

Problem: $-(-4)$

Solution:

4

Exercise:

Problem: $-(-8)$

Exercise:

Problem: $-(-15)$

Solution:

15

Exercise:

Problem: $-(-11)$

In the following exercises, evaluate.

Exercise:

Problem: $-m$ when

ⓐ $m = 3$
ⓑ $m = -3$

Solution:

ⓐ −3
ⓑ 3

Exercise:

Problem: $-p$ when

ⓐ $p=6$
ⓑ $p=-6$

Exercise:

Problem: $-c$ when

ⓐ $c=12$
ⓑ $c=-12$

Solution:

ⓐ −12;
ⓑ 12

Exercise:

Problem: $-d$ when

ⓐ $d=21$
ⓑ $d=-21$

Simplify Expressions with Absolute Value

In the following exercises, simplify each absolute value expression.
Exercise:

Problem:

ⓐ $|7|$
ⓑ $|-25|$

ⓒ $|0|$

Solution:

ⓐ 7
ⓑ 25
ⓒ 0

Exercise:

Problem:

ⓐ $|5|$
ⓑ $|20|$
ⓒ $|-19|$

Exercise:

Problem:

ⓐ $|-32|$
ⓑ $|-18|$
ⓒ $|16|$

Solution:

ⓐ 32
ⓑ 18
ⓒ 16

Exercise:

Problem:

ⓐ $|-41|$
ⓑ $|-40|$

ⓒ $|22|$

In the following exercises, evaluate each absolute value expression.

Exercise:

Problem:

ⓐ $|x|$ when $x=-28$
ⓑ $|-u|$ when $u=-15$

Solution:

ⓐ 28
ⓑ 15

Exercise:

Problem:

ⓐ $|y|$ when $y=-37$
ⓑ $|-z|$ when $z=-24$

Exercise:

Problem:

ⓐ $-|p|$ when $p=19$
ⓑ $-|q|$ when $q=-33$

Solution:

ⓐ −19
ⓑ −33

Exercise:

Problem:

ⓐ $-|a|$ when $a = 60$
ⓑ $-|b|$ when $b = -12$

In the following exercises, fill in $<, >,$ or $=$ to compare each expression.
Exercise:

Problem:

ⓐ $-6\_\_|-6|$
ⓑ $-|-3|\_\_-3$

Solution:

ⓐ <
ⓑ =

Exercise:

Problem:

ⓐ $-8\_\_|-8|$
ⓑ $-|-2|\_\_-2$

Exercise:

Problem:

ⓐ $|-3|\_\_-|-3|$
ⓑ $4\_\_-|-4|$

Solution:

ⓐ >

ⓑ $>$

Exercise:

Problem:

ⓐ $|-5|\_\_ - |-5|$
ⓑ $9\_\_ - |-9|$

In the following exercises, simplify each expression.

Exercise:

Problem: $|8-4|$

Solution:

4

Exercise:

Problem: $|9-6|$

Exercise:

Problem: $8|-7|$

Solution:

56

Exercise:

Problem: $5|-5|$

Exercise:

Problem: $|15-7|-|14-6|$

Solution:

0

Exercise:

Problem: $|17-8|-|13-4|$

Exercise:

Problem: $18-|2(8-3)|$

Solution:

8

Exercise:

Problem: $15-|3(8-5)|$

Exercise:

Problem: $8(14-2|-2|)$

Solution:

80

Exercise:

Problem: $6(13-4|-2|)$

Translate Word Phrases into Expressions with Integers

Translate each phrase into an expression with integers. *Do not simplify*.

Exercise:

Problem:

ⓐ the opposite of 8
ⓑ the opposite of -6
ⓒ negative three
ⓓ 4 minus negative 3

Solution:

ⓐ -8
ⓑ $-(-6)$, or 6
ⓒ -3
ⓓ $4-(-3)$

Exercise:

Problem:

ⓐ the opposite of 11
ⓑ the opposite of -4
ⓒ negative nine
ⓓ 8 minus negative 2

Exercise:

Problem:

ⓐ the opposite of 20
ⓑ the opposite of -5
ⓒ negative twelve
ⓓ 18 minus negative 7

Solution:

ⓐ -20
ⓑ $-(-5)$, or 5
ⓒ -12
ⓓ $18-(-7)$

Exercise:

Problem:

ⓐ the opposite of 15
ⓑ the opposite of -9
ⓒ negative sixty
ⓓ 12 minus 5

Exercise:

Problem: a temperature of 6 degrees below zero

Solution:

-6 degrees

Exercise:

Problem: a temperature of 14 degrees below zero

Exercise:

Problem: an elevation of 40 feet below sea level

Solution:

-40 feet

Exercise:

Problem: an elevation of 65 feet below sea level

Exercise:

Problem: a football play loss of 12 yards

Solution:

−12 yards

Exercise:

Problem: a football play gain of 4 yards

Exercise:

Problem: a stock gain of $3

Solution:

$3

Exercise:

Problem: a stock loss of $5

Exercise:

Problem: a golf score one above par

Solution:

+1

Exercise:

Problem: a golf score of 3 below par

Everyday Math

Exercise:

Problem:

Elevation The highest elevation in the United States is Mount McKinley, Alaska, at $20,320$ feet above sea level. The lowest elevation is Death Valley, California, at 282 feet below sea level. Use integers to write the elevation of:

ⓐ Mount McKinley
ⓑ Death Valley

Solution:

ⓐ 20,320 feet
ⓑ −282 feet

Exercise:

Problem:

Extreme temperatures The highest recorded temperature on Earth is 58° Celsius, recorded in the Sahara Desert in 1922. The lowest recorded temperature is 90° below 0° Celsius, recorded in Antarctica in 1983. Use integers to write the:

ⓐ highest recorded temperature
ⓑ lowest recorded temperature

Exercise:

Problem:

State budgets In June, 2011, the state of Pennsylvania estimated it would have a budget surplus of $\$540$ million. That same month, Texas estimated it would have a budget deficit of $\$27$ billion. Use integers to write the budget:

ⓐ surplus

ⓑ deficit

Solution:

ⓐ \$540 million
ⓑ −\$27 billion

Exercise:

Problem:

College enrollments Across the United States, community college enrollment grew by 1,400,000 students from 2007 to 2010. In California, community college enrollment declined by 110,171 students from 2009 to 2010. Use integers to write the change in enrollment:

ⓐ growth
ⓑ decline

Writing Exercises

Exercise:

Problem:

Give an example of a negative number from your life experience.

Solution:

Sample answer: I have experienced negative temperatures.

Exercise:

Problem:

What are the three uses of the “−” sign in algebra? Explain how they differ.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
locate positive and negative numbers on the number line.			
order positive and negative numbers.			
find opposites.			
simplify expressions with absolute value.			
translate word phrases to expressions with integers.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

absolute value
: The absolute value of a number is its distance from 0 on the number line.

integers
: Integers are counting numbers, their opposites, and zero ... –3, –2, –1, 0, 1, 2, 3 ...

negative number
: A negative number is less than zero.

opposites
: The opposite of a number is the number that is the same distance from zero on the number line, but on the opposite side of zero.

Add Integers
By the end of this section, you will be able to:

- Model addition of integers
- Simplify expressions with integers
- Evaluate variable expressions with integers
- Translate word phrases to algebraic expressions
- Add integers in applications

Note:
Before you get started, take this readiness quiz.

1. Evaluate $x+8$ when $x=6$.
   If you missed this problem, review [link].
2. Simplify: $8+2\left(5+1\right)$.
   If you missed this problem, review [link].
3. Translate *the sum of 3 and negative 7* into an algebraic expression.
   If you missed this problem, review [link]

Model Addition of Integers

Now that we have located positive and negative numbers on the number line, it is time to discuss arithmetic operations with integers.

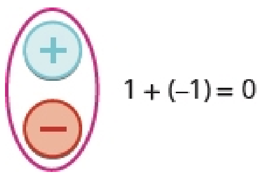
Most students are comfortable with the addition and subtraction facts for positive numbers. But doing addition or subtraction with both positive and negative numbers may be more difficult. This difficulty relates to the way the brain learns.

The brain learns best by working with objects in the real world and then generalizing to abstract concepts. Toddlers learn quickly that if they have two cookies and their older brother steals one, they have only one left. This is a concrete example of $2-1$. Children learn their basic addition and subtraction facts from experiences in their everyday lives. Eventually, they know the number facts without relying on cookies.

Addition and subtraction of negative numbers have fewer real world examples that are meaningful to us. Math teachers have several different approaches, such as number lines, banking, temperatures, and so on, to make these concepts real.

We will model addition and subtraction of negatives with two color counters. We let a blue counter represent a positive and a red counter will represent a negative. To make this easier to read when printed without color, we have placed a "+" in the positive counters and a "-" in the negative counters.

If we have one positive and one negative counter, the value of the pair is zero. They form a neutral pair. The value of this neutral pair is zero as summarized in [link].



A blue counter represents $+1$. A red counter represents -1. Together they add to zero.

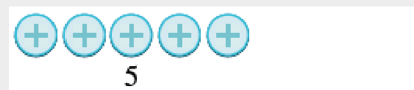
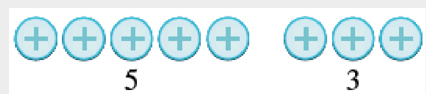
We will model four addition facts using the numbers $5, -5$ and 3 , -3.
Equation:

$$5+3 \quad -5+(-3) \quad -5+3 \quad 5+(-3)$$

Example:
Exercise:

Problem: Model: $5+3$.

Solution:
Solution

Interpret the expression.	$5+3$ means the sum of 5 and 3.
Model the first number. Start with 5 positives.	
Model the second number. Add 3 positives.	
Count the total number of counters.	
The sum of 5 and 3 is 8.	$5+3=8$

Note:
Exercise:

Problem: Model the expression.

$2+4$

Solution:

6

Note:
Exercise:

Problem: Model the expression.

$2+5$

Solution:

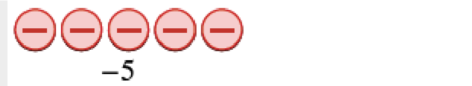
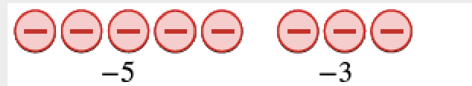
7

Example:
Exercise:

Problem: Model: $-5+(-3)$.

Solution:
Solution

Interpret the expression.	$-5+(-3)$ means the sum of -5 and -3.
Model the first number. Start with 5	

negatives.	
Model the second number. Add 3 negatives.	
Count the total number of counters.	
The sum of −5 and −3 is −8.	$-5 + -3 = -8$

Note:
Exercise:

Problem: Model the expression.

$-2 + (-4)$

Solution:

−6

Note:
Exercise:

Problem: Model the expression.

$-2+(-5)$

Solution:

-7

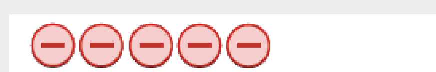
[link] and [link] are very similar. The first example adds 5 positives and 3 positives—both positives. The second example adds 5 negatives and 3 negatives—both negatives. In each case, we got a result of 8—either 8 positives or 8 negatives. When the signs are the same, the counters are all the same color.

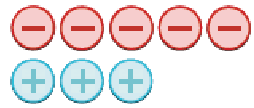
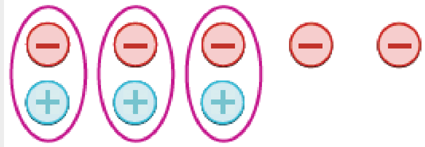
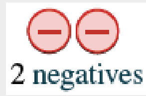
Now let's see what happens when the signs are different.

Example:
Exercise:

Problem: Model: $-5+3$.

Solution:
Solution

Interpret the expression.	$-5+3$ means the sum of -5 and 3.
Model the first number. Start with 5 negatives.	
Model the second number. Add 3	

positives.	
Remove any neutral pairs.	
Count the result.	
The sum of −5 and 3 is −2.	$-5+3=-2$

Notice that there were more negatives than positives, so the result is negative.

Note:
Exercise:

Problem: Model the expression, and then simplify:

$2+(-4)$

Solution:

−2

Note:
Exercise:

Problem: Model the expression, and then simplify:

$2 + (-5)$

Solution:

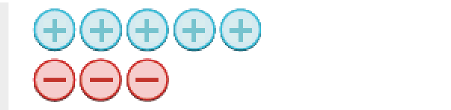
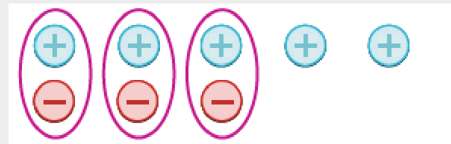
−3

Example:
Exercise:

Problem: Model: $5 + (-3)$.

Solution:
Solution

Interpret the expression.	$5 + (-3)$ means the sum of 5 and -3.
Model the first number. Start with 5 positives.	
Model the second number. Add 3 negatives.	

	
Remove any neutral pairs.	
Count the result.	
The sum of 5 and −3 is 2.	$5 + (-3) = 2$

Note:
Exercise:

Problem: Model the expression, and then simplify:

$(-2) + 4$

Solution:

2

Note:

Exercise:

Problem: Model the expression:

$(-2) + 5$

Solution:

3

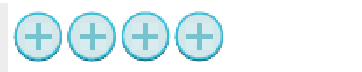
Example:
Modeling Addition of Positive and Negative Integers
Model each addition.
Exercise:

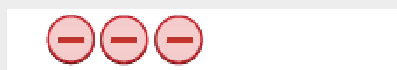
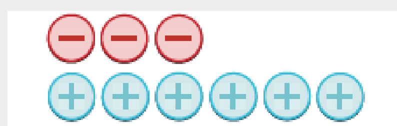
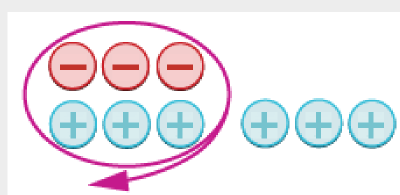
Problem:

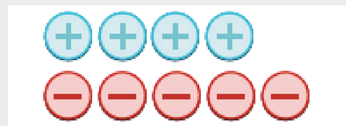
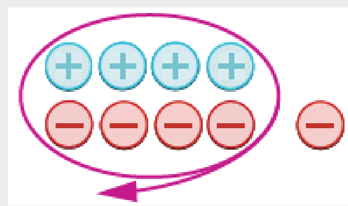
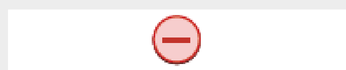
ⓐ 4 + 2
ⓑ −3 + 6
ⓒ 4 + (−5)
ⓓ -2 + (−3)

Solution:

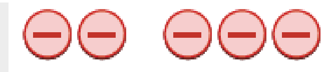
ⓐ	
	$4 + 2$
Start with 4 positives.	

	
Add two positives.	
How many do you have?	6. $4+2=6$

ⓑ	
	$-3+6$
Start with 3 negatives.	
Add 6 positives.	
Remove neutral pairs.	
How many are left?	
	3. $-3+6=3$

ⓒ	
	$4 + (-5)$
Start with 4 positives.	
Add 5 negatives.	
Remove neutral pairs.	
How many are left?	
	$-1.\ 4 + (-5) = -1$

ⓓ	
	$-2 + (-3)$
Start with 2 negatives.	

Add 3 negatives.	
How many do you have?	-5. $-2+(-3)=-5$

Note:
Exercise:

Problem: Model each addition.

ⓐ 3 + 4
ⓑ −1 + 4
ⓒ 4 + (−6)
ⓓ −2 + (−2)

Solution:

ⓐ

ⓑ

ⓒ

ⓓ

Note:

Exercise:

Problem:

ⓐ 5 + 1
ⓑ −3 + 7
ⓒ 2 + (−8)
ⓓ −3 + (−4)

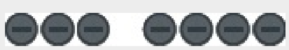
Solution:

ⓐ

ⓑ

ⓒ

ⓓ



Properties of Adding Integers

You should recall that we noticed some properties for adding whole numbers. Since whole numbers are also integers we want to make sure that those properties haven't been lost when we think of whole numbers as integers and to see if they apply for all integers as well.

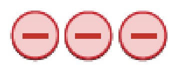
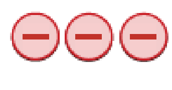
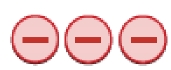
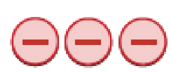
Identity Property of Addition

The sum of any number a and 0 is the number:

$a + 0 = a$
$0 + a = a$

For example, -3 + 0 = -3.
Why is the property true?

Adding zero does not change the counters that we already had.

	$-3 + 0$
Start with 3 negatives.	
Add nothing.	
Remove neutral pairs.	 (There are none to remove.)
How many are left?	
	-3. $-3 + 0 = -3$

Addition Property of Opposites
The sum of any number and its opposite is 0:
$a + (\text{-}a) = 0$

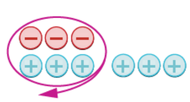
Any number combined with its opposite will form the same number of neutral pairs, with nothing left over.

Commutative Property of Addition
Changing the order of the addends a and b does not change their sum.

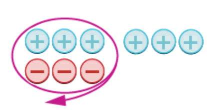
$$a + b = b + a$$

For example, (-3) + 6 = 6 + (-3)?
Why is the property true?

The order the counters are entered makes no difference. Three pairs of positive and negative counters form neutral pairs either way.

	$-3+6$
Start with 3 negatives.	
Add 6 positives.	
Remove neutral pairs.	
How many are left?	

=

	$6+(-3)$
Start with 6 positives.	
Add 3 negatives.	
Remove neutral pairs.	
How many are left?	

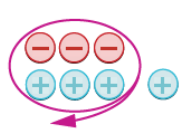
	3. $-3 + 6 = 3$

	3. $6 + (-3) = 3$

The Associative Property of Addition
Changing the grouping of the addends a, b, and c does not change the sum.

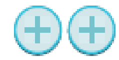
$(\mathrm{a} + \mathrm{b}) + c = a + (\mathrm{b} + \mathrm{c})$

For example, (-3 + 4) + (-2) = -3 + (4 + (-2))?
Why is the property true?

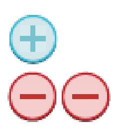
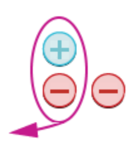
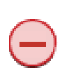
	$-3 + 4$
Start with 3 negatives.	
Add 4 positives.	
Remove neutral pairs.	
How	

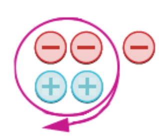
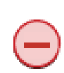
	$4 + (-2)$
Start with 4 positives.	
Add 2 negatives.	
Remove neutral pairs.	

many are left?	
	1. $-3+4=1$

How many are left?	
	2. $4+(-2)=2$

The computations continue with the next set of additions. The sums in the first part become addends in the second part.

	$1+(-2)$
Start with 1 positive.	
Add 2 negatives.	
Remove neutral pairs.	
How many are	

	$-3+2$
Start with 3 negatives.	
Add 2 positives.	
Remove neutral pairs.	
How many are left?	

left?	
	-1. $1 + \text{(-2)} = -1$

	-1. $-3 + 2 = -1$

It is only at the very end that the two computations have to be equal.
(-3 + 4) + (-2) = -3 + (4 + (-2)) because both equal -1.

Even when there are more than two addends it makes no difference how they are grouped for computing the final sum.

Summary of Addition Properties
The identity, commutative and associative properties of addition are true for integers.

Simplify Expressions with Integers

Now that you have modeled adding small positive and negative integers, you can visualize the model in your mind to simplify expressions with any integers.

For example, if you want to add $37 + (-53)$, you don't have to count out 37 blue counters and 53 red counters.

Picture 37 blue counters with 53 red counters lined up underneath. Since there would be more negative counters than positive counters, the sum would be negative. Because $53-37 = 16$, there are 16 more negative counters.
Equation:

$$37 + (-53) = -16$$

Let's try another one. We'll add $-74 + (-27)$. Imagine 74 red counters and 27 more red counters, so we have 101 red counters all together. This means the sum is -101.
Equation:

$$-74 + (-27) = -101$$

Look again at the results of [link] - [link].

$5+3$	$-5+(-3)$
both positive, sum positive	both negative, sum negative
When the signs are the same, the counters would be all the same color, so add them.	
$-5+3$	$5+(-3)$
different signs, more negatives	different signs, more positives
Sum negative	sum positive
When the signs are different, some counters would make neutral pairs; subtract to see how many are left.	

Addition of Positive and Negative Integers

Example:
Exercise:

Problem: Simplify:

ⓐ $19+(-47)$
ⓑ $-32+40$

Solution:
Solution

ⓐ Since the signs are different, we subtract 19 from 47. The answer will be negative because there are more negatives than positives.
Equation:

$$19+(-47)$$
$$-28$$

ⓑ The signs are different so we subtract 32 from 40. The answer will be positive because there are more positives than negatives

Equation:

$$-32+40$$
$$8$$

Note:
Exercise:

Problem: Simplify each expression:

ⓐ $15+(-32)$
ⓑ $-19+76$

Solution:

ⓐ −17
ⓑ 57

Note:
Exercise:

Problem: Simplify each expression:

ⓐ $-55+9$
ⓑ $43+(-17)$

Solution:

ⓐ −46
ⓑ 26

Example:
Exercise:

Problem: Simplify: $-14+(-36)$.

Solution:
Solution

Since the signs are the same, we add. The answer will be negative because there are only negatives.
Equation:

$$\begin{array}{c} -14+(-36) \\ -50 \end{array}$$

Note:
Exercise:

Problem: Simplify the expression:

$-31+(-19)$

Solution:

−50

Note:
Exercise:

Problem: Simplify the expression:

$-42+(-28)$

Solution:

−70

The techniques we have used up to now extend to more complicated expressions. Remember to follow the order of operations.

Example:
Exercise:

Problem: Simplify: $-5+3(-2+7)$.

Solution:
Solution

	$-5+3(-2+7)$
Simplify inside the parentheses.	$-5+3(5)$
Multiply.	$-5+15$
Add left to right.	10

Note:
Exercise:

Problem: Simplify the expression:

$-2+5(-4+7)$

Solution:

13

Note:
Exercise:

Problem: Simplify the expression:

$-4+2(-3+5)$

Solution:

0

Evaluate Variable Expressions with Integers

Remember that to evaluate an expression means to substitute a number for the variable in the expression. Now we can use negative numbers as well as positive numbers when evaluating expressions.

Example:
Exercise:

Problem: Evaluate $x+7$ when

ⓐ $x=-2$
ⓑ $x=-11$.

Solution:
Solution

ⓐ Evaluate $x+7$ when $x=-2$	
	$-2+7$

Substitute -2 for x.	$-2+7$
Simplify.	5

ⓑ Evaluate $x+7$ when $x=-11$	
	$x+7$
Substitute -11 for x.	$-11+7$
Simplify.	-4

Note:
Exercise:

Problem: Evaluate each expression for the given values:

$x+5$ when

ⓐ $x=-3$ and
ⓑ $x=-17$

Solution:

ⓐ 2
ⓑ −12

Note:
Exercise:

Problem: Evaluate each expression for the given values: $y + 7$ when

ⓐ $y = -5$
ⓑ $y = -8$

Solution:

ⓐ 2
ⓑ −1

Example:
Exercise:

Problem: When $n = -5$, evaluate

ⓐ $n + 1$
ⓑ $-n + 1.$

Solution:
Solution

ⓐ Evaluate $n+1$ when $n=-5$	
	$n+1$
Substitute -5 for n.	$-5+1$
Simplify.	-4

ⓑ Evaluate $-n+1$ when $n=-5$	
	$-n+1$
Substitute -5 for n.	$-(-5)+1$
Simplify.	$5+1$
Add.	6

Note:
Exercise:

Problem: When $n = -8$, evaluate

ⓐ $n + 2$
ⓑ $-n + 2$

Solution:

ⓐ −6
ⓑ 10

Note:
Exercise:

Problem: When $y = -9,$ evaluate

ⓐ $y + 8$
ⓑ $-y + 8.$

Solution:

ⓐ −1
ⓑ 17

Next we'll evaluate an expression with two variables.

Example:
Exercise:

Problem: Evaluate $3a + b$ when $a = 12$ and $b = -30.$

Solution:

Solution

	$3a + b$
Substitute 12 for a and -30 for b.	$3(12) + (-30)$
Multiply.	$36 + (-30)$
Add.	6

Note:
Exercise:

Problem: Evaluate the expression:

$a + 2b$ when $a = -19$ and $b = 14$.

Solution:

9

Note:
Exercise:

Problem: Evaluate the expression:

$5p + q$ when $p = 4$ and $q = -7$.

Solution:

13

Example:
Exercise:

Problem: Evaluate $(x + y)^2$ when $x = -18$ and $y = 24$.

Solution:
Solution

This expression has two variables. Substitute -18 for x and 24 for y.

	$(x + y)^2$
Substitute -18 for x and 24 for y.	$(-18 + 24)^2$
Add inside the parentheses.	$(6)^2$
Simplify	36

Note:
Exercise:

Problem: Evaluate:

$(x+y)^2$ when $x=-15$ and $y=29$.

Solution:

196

Note:
Exercise:

Problem: Evaluate:

$(x+y)^3$ when $x=-8$ and $y=10$.

Solution:

8

Translate Word Phrases to Algebraic Expressions

All our earlier work translating word phrases to algebra also applies to expressions that include both positive and negative numbers. Remember that the phrase *the sum* indicates addition.

Example:
Exercise:

Problem: Translate and simplify: the sum of -9 and 5.

Solution:
Solution

The sum of −9 and 5 indicates addition.	the sum of -9 and 5
Translate.	$-9+5$
Simplify.	-4

Note:
Exercise:

Problem: Translate and simplify the expression:

the sum of -7 and 4

Solution:

−7 + 4 = −3

Note:
Exercise:

Problem: Translate and simplify the expression:

the sum of -8 and -6

Solution:

−8 + (−6) = −14

Example:
Exercise:

Problem: Translate and simplify: the sum of 8 and -12, increased by 3.

Solution:
Solution

The phrase *increased by* indicates addition.

	The sum of 8 and -12, increased by 3
Translate.	$[8+(-12)]+3$
Simplify.	$-4+3$
Add.	-1

Note:
Exercise:

Problem: Translate and simplify:

the sum of 9 and -16, increased by 4.

Solution:

[9 + (−16)] + 4 = −3

Note:
Exercise:

Problem: Translate and simplify:

the sum of -8 and -12, increased by 7.

Solution:

$[-8 + (-12)] + 7 = -13$

Add Integers in Applications

Recall that we were introduced to some situations in everyday life that use positive and negative numbers, such as temperatures, banking, and sports. For example, a debt of $\$5$ could be represented as $-\$5$. Let's practice translating and solving a few applications.

Solving applications is easy if we have a plan. First, we determine what we are looking for. Then we write a phrase that gives the information to find it. We translate the phrase into math notation and then simplify to get the answer. Finally, we write a sentence to answer the question.

Example:
Exercise:

Problem:

The temperature in Buffalo, NY, one morning started at 7 degrees below zero Fahrenheit. By noon, it had warmed up 12 degrees. What was the temperature at noon?

Solution:
Solution

We are asked to find the temperature at noon.

Write a phrase for the temperature.	The temperature warmed up 12 degrees from 7 degrees below zero.
Translate to math	$-7 + 12$

notation.	
Simplify.	5
Write a sentence to answer the question.	The temperature at noon was 5 degrees Fahrenheit.

Note:
Exercise:

Problem:

The temperature in Chicago at 5 A.M. was 10 degrees below zero Celsius. Six hours later, it had warmed up 14 degrees Celsius. What is the temperature at 11 A.M.?

Solution:

4 degrees Celsius

Note:
Exercise:

Problem:

A scuba diver was swimming 16 feet below the surface and then dove down another 17 feet. What is her new depth?

Solution:

-33 feet

Example:
Exercise:

Problem:

A football team took possession of the football on their 42-yard line. In the next three plays, they lost 6 yards, gained 4 yards, and then lost 8 yards. On what yard line was the ball at the end of those three plays?

Solution:
Solution

We are asked to find the yard line the ball was on at the end of three plays.

Write a word phrase for the position of the ball.	Start at 42, then lose 6, gain 4, lose 8.
Translate to math notation.	$42 - 6 + 4 - 8$
Simplify.	32
Write a sentence to answer the question.	At the end of the three plays, the ball is on the 32-yard line.

Note:
Exercise:

Problem:

The Bears took possession of the football on their 20-yard line. In the next three plays, they lost 9 yards, gained 7 yards, then lost 4 yards. On what yard line was the ball at the end of those three plays?

Solution:

14-yard line

Note:
Exercise:

Problem:

The Chargers began with the football on their 25-yard line. They gained 5 yards, lost 8 yards and then gained 15 yards on the next three plays. Where was the ball at the end of these plays?

Solution:

37-yard line

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Adding Integers with Same Sign Using Color Counters
- Adding Integers with Different Signs Using Counters
- Ex1: Adding Integers
- Ex2: Adding Integers

Key Concepts

- **Addition of Positive and Negative Integers**

$5+3$	$-5+(-3)$
both positive, sum positive	both negative, sum negative
When the signs are the same, the counters would be all the same color, so add them.	
$-5+3$	$5+(-3)$

different signs, more negatives	different signs, more positives
Sum negative	sum positive
When the signs are different, some counters would make neutral pairs; subtract to see how many are left.	

Properties of Addition
The Identity, Commutative, and Associative Properties of Addition are true for Integers.

Practice Makes Perfect

Model Addition of Integers

In the following exercises, model the expression to simplify.

Exercise:

Problem: $7 + 4$

Solution:

11

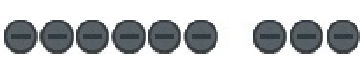
Exercise:

Problem: $8 + 5$

Exercise:

Problem: $-6 + (-3)$

Solution:



-9

Exercise:

Problem: $-5 + (-5)$

Exercise:

Problem: $-7 + 5$

Solution:

-2

Exercise:

Problem: $-9 + 6$

Exercise:

Problem: $8 + (-7)$

Solution:

1

Exercise:

Problem: $9 + (-4)$

Simplify Expressions with Integers

In the following exercises, simplify each expression.

Exercise:

Problem: $-21 + (-59)$

Solution:

−80

Exercise:

Problem: $-35 + (-47)$

Exercise:

Problem: $48 + (-16)$

Solution:

32

Exercise:

Problem: $34 + (-19)$

Exercise:

Problem: $-200 + 65$

Solution:

−135

Exercise:

Problem: $-150 + 45$

Exercise:

Problem: $2 + (-8) + 6$

Solution:

0

Exercise:

Problem: $4+(-9)+7$

Exercise:

Problem: $-14+(-12)+4$

Solution:

-22

Exercise:

Problem: $-17+(-18)+6$

Exercise:

Problem: $135+(-110)+83$

Solution:

108

Exercise:

Problem: $140+(-75)+67$

Exercise:

Problem: $-32+24+(-6)+10$

Solution:

-4

Exercise:

Problem: $-38+27+(-8)+12$

Exercise:

Problem: $19+2(-3+8)$

Solution:

29

Exercise:

Problem: $24 + 3(-5 + 9)$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

Exercise:

Problem: $x + 8$ when

ⓐ $x = -26$
ⓑ $x = -95$

Solution:

ⓐ −18
ⓑ −87

Exercise:

Problem: $y + 9$ when

ⓐ $y = -29$
ⓑ $y = -84$

Exercise:

Problem: $y + (-14)$ when

ⓐ $y = -33$
ⓑ $y = 30$

Solution:

ⓐ −47
ⓑ 16

Exercise:

Problem: $x + (-21)$ when

ⓐ $x = -27$
ⓑ $x = 44$

Exercise:

Problem: When $a = -7$, evaluate:

ⓐ $a + 3$
ⓑ $-a + 3$

Solution:

ⓐ −4
ⓑ 10

Exercise:

Problem: When $b = -11$, evaluate:

ⓐ $b + 6$
ⓑ $-b + 6$

Exercise:

Problem: When $c = -9$, evaluate:

ⓐ $c + (-4)$
ⓑ $-c + (-4)$

Solution:

ⓐ −13

ⓑ 5

Exercise:

Problem: When $d = -8$, evaluate:

ⓐ $d + (-9)$
ⓑ $-d + (-9)$

Exercise:

Problem: $m + n$ when, $m = -15$, $n = 7$

Solution:

−8

Exercise:

Problem: $p + q$ when, $p = -9$, $q = 17$

Exercise:

Problem: $r - 3s$ when, $r = 16$, $s = 2$

Solution:

10

Exercise:

Problem: $2t + u$ when, $t = -6$, $u = -5$

Exercise:

Problem: $(a + b)^2$ when, $a = -7$, $b = 15$

Solution:

64

Exercise:

Problem: $(c + d)^2$ when, $c = -5$, $d = 14$

Exercise:

Problem: $(x + y)^2$ when, $x = -3$, $y = 14$

Solution:

121

Exercise:

Problem: $(y + z)^2$ when, $y = -3$, $z = 15$

Translate Word Phrases to Algebraic Expressions

In the following exercises, translate each phrase into an algebraic expression and then simplify.

Exercise:

Problem: The sum of -14 and 5

Solution:

$-14 + 5 = -9$

Exercise:

Problem: The sum of -22 and 9

Exercise:

Problem: 8 more than -2

Solution:

$-2 + 8 = 6$

Exercise:

Problem: 5 more than -1

Exercise:

Problem: -10 added to -15

Solution:

$-15 + (-10) = -25$

Exercise:

Problem: -6 added to -20

Exercise:

Problem: 6 more than the sum of -1 and -12

Solution:

$[-1 + (-12)] + 6 = -7$

Exercise:

Problem: 3 more than the sum of -2 and -8

Exercise:

Problem: the sum of 10 and -19, increased by 4

Solution:

$[10 + (-19)] + 4 = -5$

Exercise:

Problem: the sum of 12 and -15, increased by 1

Add Integers in Applications

In the following exercises, solve.

Exercise:

Problem:

Temperature The temperature in St. Paul, Minnesota was $-19^\circ F$ at sunrise. By noon the temperature had risen $26^\circ F$. What was the temperature at noon?

Solution:

7°F

Exercise:

Problem:

Temperature The temperature in Chicago was $-15^\circ F$ at 6 am. By afternoon the temperature had risen $28^\circ F$. What was the afternoon temperature?

Exercise:

Problem:

Credit Cards Lupe owes $\$73$ on her credit card. Then she charges $\$45$ more. What is the new balance?

Solution:

−\$118

Exercise:

Problem:

Credit Cards Frank owes $\$212$ on his credit card. Then he charges $\$105$ more. What is the new balance?

Exercise:

Problem:

Weight Loss Angie lost 3 pounds the first week of her diet. Over the next three weeks, she lost 2 pounds, gained 1 pound, and then lost 4 pounds. What was the change in her weight over the four weeks?

Solution:

−8 pounds

Exercise:

Problem:

Weight Loss April lost 5 pounds the first week of her diet. Over the next three weeks, she lost 3 pounds, gained 2 pounds, and then lost 1 pound. What was the change in her weight over the four weeks?

Exercise:

Problem:

Football The Rams took possession of the football on their own 35-yard line. In the next three plays, they lost 12 yards, gained 8 yards, then lost 6 yards. On what yard line was the ball at the end of those three plays?

Solution:

25-yard line

Exercise:

Problem:

Football The Cowboys began with the ball on their own 20-yard line. They gained 15 yards, lost 3 yards and then gained 6 yards on the next three plays. Where was the ball at the end of these plays?

Exercise:

Problem:

Calories Lisbeth walked from her house to get a frozen yogurt, and then she walked home. By walking for a total of 20 minutes, she burned 90 calories. The frozen yogurt she ate was 110 calories. What was her total calorie gain or loss?

Solution:

20 calories

Exercise:

Problem:

Calories Ozzie rode his bike for 30 minutes, burning 168 calories. Then he had a 140-calorie iced blended mocha. Represent the change in calories as an integer.

Everyday Math

Exercise:

Problem:

Stock Market The week of September 15, 2008, was one of the most volatile weeks ever for the U.S. stock market. The change in the Dow Jones Industrial Average each day was:

$$\begin{array}{llllll} \text{Monday} & -504 & \text{Tuesday} & +142 & \text{Wednesday} & -449 \\ \text{Thursday} & +410 & \text{Friday} & +369 & & \end{array}$$

What was the overall change for the week?

Solution:

-32

Exercise:

Problem:

Stock Market During the week of June 22, 2009, the change in the Dow Jones Industrial Average each day was:

$$\begin{array}{llllll} \text{Monday} & -201 & \text{Tuesday} & -16 & \text{Wednesday} & -23 \\ \text{Thursday} & +172 & \text{Friday} & -34 & & \end{array}$$

What was the overall change for the week?

Writing Exercises

Exercise:

Problem:

Explain why the sum of -8 and 2 is negative, but the sum of 8 and -2 and is positive.

Solution:

Sample answer: In the first case, there are more negatives so the sum is negative. In the second case, there are more positives so the sum is positive.

Exercise:

Problem:

Give an example from your life experience of adding two negative numbers.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
model addition of integers.			
simplify expressions with integers.			
evaluate variable expressions with integers.			
translate word phrases to algebraic expressions.			
add integers in applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Subtract Integers
By the end of this section, you will be able to:

- Model subtraction of integers
- Simplify expressions with integers
- Evaluate variable expressions with integers
- Translate words phrases to algebraic expressions
- Subtract integers in applications

Note:
Before you get started, take this readiness quiz.

1. Simplify: $12-(8-1)$.
   If you missed this problem, review [link].
2. Translate *the difference of 20 and -15* into an algebraic expression.
   If you missed this problem, review [link].
3. Add: $-18+7$.
   If you missed this problem, review [link].

Model Subtraction of Integers

Remember the story in the last section about the toddler and the cookies? Children learn how to subtract numbers through their everyday experiences. Real-life experiences serve as models for subtracting positive numbers, and in some cases, such as temperature, for adding negative as well as positive numbers. But it is difficult to relate subtracting negative numbers to common life experiences. Most people do not have an intuitive understanding of subtraction when negative numbers are involved. Math teachers use several different models to explain subtracting negative numbers.

We will continue to use counters to model subtraction. Remember, the blue counters represent positive numbers and the red counters represent negative

numbers.

Perhaps when you were younger, you read $5-3$ as *five take away three*. When we use counters, we can think of subtraction the same way.

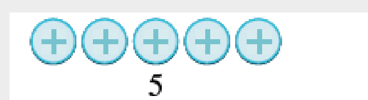
We will model four subtraction facts using the numbers 5 and 3.
Equation:

$$5-3 \quad -5-(-3) \quad -5-3 \quad 5-(-3)$$

Example:
Exercise:

Problem: Model: $5-3$.

Solution:
Solution

Interpret the expression.	$5-3$ means 5 take away 3.
Model the first number. Start with 5 positives.	
Take away the second number. So take away 3 positives.	

Find the counters that are left.	
	$5 - 3 = 2$. The difference between 5 and 3 is 2.

Note:
Exercise:

Problem: Model the expression:

$6 - 4$

Solution:

2

Note:
Exercise:

Problem: Model the expression:

$7 - 4$

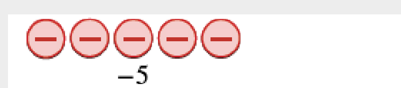
Solution:

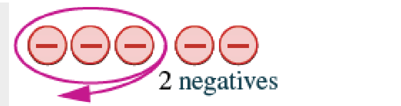
3

Example:
Exercise:

Problem: Model: $-5 - (-3)$.

Solution:
Solution

Interpret the expression.	$-5 - (-3)$ means -5 take away -3.
Model the first number. Start with 5 negatives.	
Take away the second number. So take away 3 negatives.	

	
Find the number of counters that are left.	
	$-5-(-3)=-2$. The difference between -5 and -3 is -2.

Note:
Exercise:

Problem: Model the expression:

$-6-(-4)$

Solution:

-2

Note:

Exercise:

Problem: Model the expression:

$-7 - (-4)$

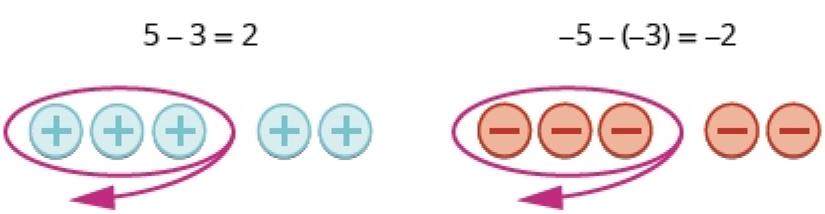
Solution:

-3

Notice that [link] and [link] are very much alike.

- First, we subtracted 3 positives from 5 positives to get 2 positives.
- Then we subtracted 3 negatives from 5 negatives to get 2 negatives.

Each example used counters of only one color, and the “take away” model of subtraction was easy to apply.

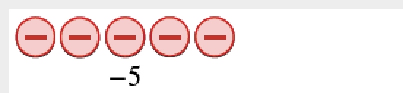
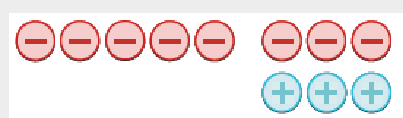
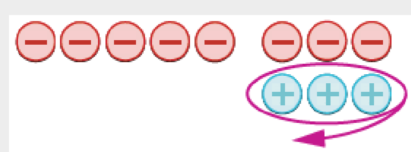


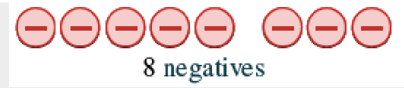
Now let’s see what happens when we subtract one positive and one negative number. We will need to use both positive and negative counters and sometimes some neutral pairs, too. Adding a neutral pair does not change the value.

Example:
Exercise:

Problem: Model: $-5 - 3$.

Solution:
Solution

Interpret the expression.	$-5 - 3$ means -5 take away 3.
Model the first number. Start with 5 negatives.	
Take away the second number. So we need to take away 3 positives.	
But there are no positives to take away. Add neutral pairs until you have 3 positives.	
Now take away 3 positives.	
Count the number of counters that	

are left.	
	$-5-3=-8$. The difference of -5 and 3 is -8.

Note:
Exercise:

Problem: Model the expression:

$-6-4$

Solution:

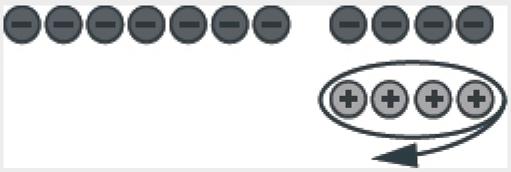
-10

Note:
Exercise:

Problem: Model the expression:

$-7 - 4$

Solution:



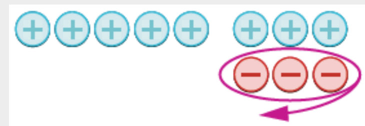
-11

Example:
Exercise:

Problem: Model: $5 - (-3)$.

Solution:
Solution

Interpret the expression.	$5 - (-3)$ means 5 take away -3.
Model the first number. Start with 5 positives.	
Take away the second number, so take away 3 negatives.	

But there are no negatives to take away. Add neutral pairs until you have 3 negatives.	
Then take away 3 negatives.	
Count the number of counters that are left.	
	The difference of 5 and -3 is 8. $5-(-3)=8$

Note:
Exercise:

Problem: Model the expression:

$6-(-4)$

Solution:

10

Note:
Exercise:

Problem: Model the expression:

$7 - (-4)$

Solution:

11

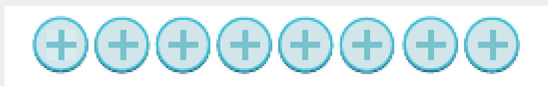
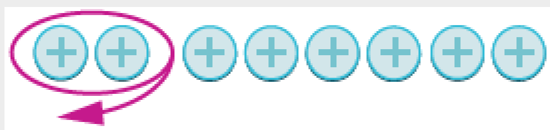
Example:
Exercise:

Problem: Model each subtraction.

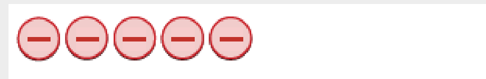
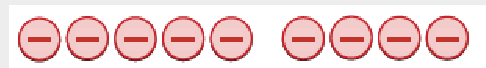
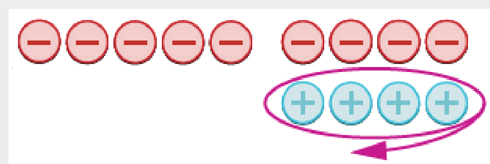
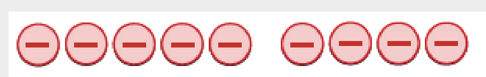
ⓐ 8 − 2
ⓑ −5 − 4

ⓒ 6 − (−6)
ⓓ −8 − (−3)

Solution:

ⓐ	
	$8 - 2$ This means 8 take away 2.
Start with 8 positives.	
Take away 2 positives.	
How many are left?	6
	$8 - 2 = 6$

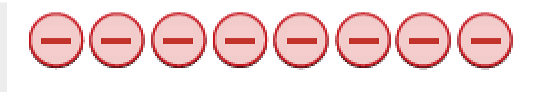
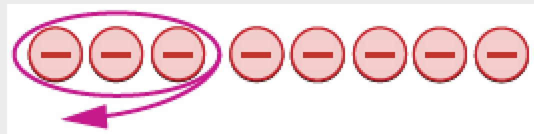
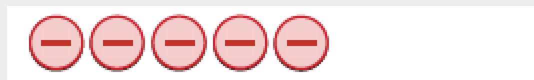
ⓑ	

	$-5-4$ This means -5 take away 4.
Start with 5 negatives.	
You need to take away 4 positives. Add 4 neutral pairs to get 4 positives.	 
Take away 4 positives.	
How many are left?	
	-9
	$-5-4=-9$

ⓒ	

	$6-(-6)$ This means 6 take away -6.
Start with 6 positives.	
Add 6 neutrals to get 6 negatives to take away.	
Remove 6 negatives.	
How many are left?	
	12
	$6-(-6)=12$

ⓓ	
	$-8-(-3)$ This means -8 take away -3.
Start with 8 negatives.	

	
Take away 3 negatives.	
How many are left?	
	-5
	$-8-(-3)=-5$

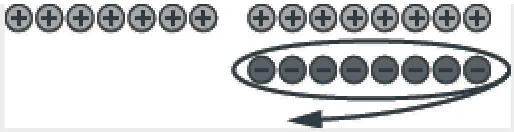
Note:
Exercise:

Problem: Model each subtraction.

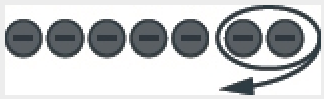
ⓐ 7 - (-8)
ⓑ -2 - (-2)
ⓒ 4 - 1
ⓓ -6 - 8

Solution:

ⓐ

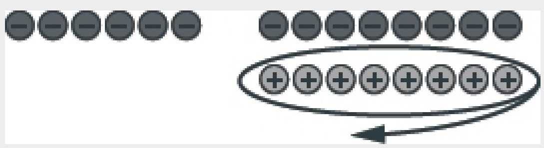


ⓑ



ⓒ

ⓓ



Note:
Exercise:

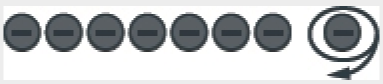
Problem: Model each subtraction.

ⓐ 4 - (-6)
ⓑ -8 - (-1)
ⓒ 7 - 3
ⓓ -4 - 2

Solution:

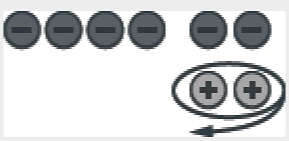
ⓐ

ⓑ



ⓒ

ⓓ



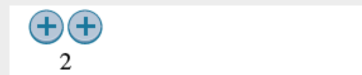
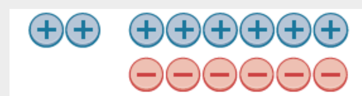
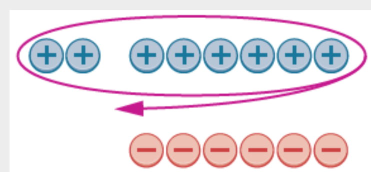
Example:
Exercise:

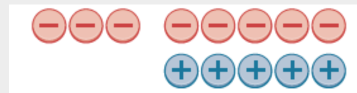
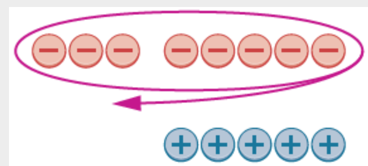
Problem: Model each subtraction expression:

ⓐ $2 - 8$
ⓑ $-3 - (-8)$

Solution:

Solution

ⓐ We start with 2 positives.	
We need to take away 8 positives, but we have only 2.	
Add neutral pairs until there are 8 positives to take away.	
Then take away eight positives.	
Find the number of counters that are left. There are 6 negatives.	
	$2 - 8 = -6$

ⓑ We start with 3 negatives.	
We need to take away 8 negatives, but we have only 3.	
Add neutral pairs until there are 8 negatives to take away.	
Then take away the 8 negatives.	
Find the number of counters that are left. There are 5 positives.	
	$-3-(-8)=5$

Note:
Exercise:

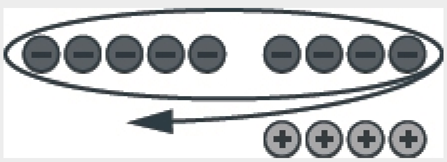
Problem: Model each subtraction expression.

ⓐ $7-9$
ⓑ $-5-(-9)$

Solution:

ⓐ

-2
ⓑ



4

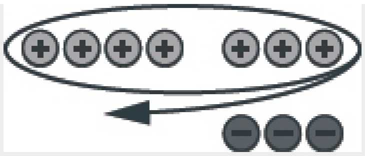
Note:
Exercise:

Problem: Model each subtraction expression.

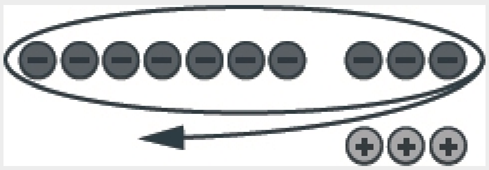
ⓐ $4 - 7$
ⓑ $-7 - (-10)$

Solution:

ⓐ



-3
ⓑ



3

Simplify Expressions with Integers

Do you see a pattern? Are you ready to subtract integers without counters? Let’s do two more subtractions. We’ll think about how we would model these with counters, but we won’t actually use the counters.

- Subtract $-23 - 7$.
  Think: We start with 23 negative counters.
  We have to subtract 7 positives, but there are no positives to take away.
  So we add 7 neutral pairs to get the 7 positives. Now we take away the 7 positives.
  So what’s left? We have the original 23 negatives plus 7 more negatives from the neutral pair. The result is 30 negatives.
  Equation:

$$-23 - 7 = -30$$

  Notice, that to subtract 7, we added 7 negatives.

- Subtract $30-(-12)$.
  Think: We start with 30 positives.
  We have to subtract 12 negatives, but there are no negatives to take away.
  So we add 12 neutral pairs to the 30 positives. Now we take away the 12 negatives.
  What's left? We have the original 30 positives plus 12 more positives from the neutral pairs. The result is 42 positives.
  Equation:

  $$30-(-12)=42$$

  Notice that to subtract -12, we added 12.

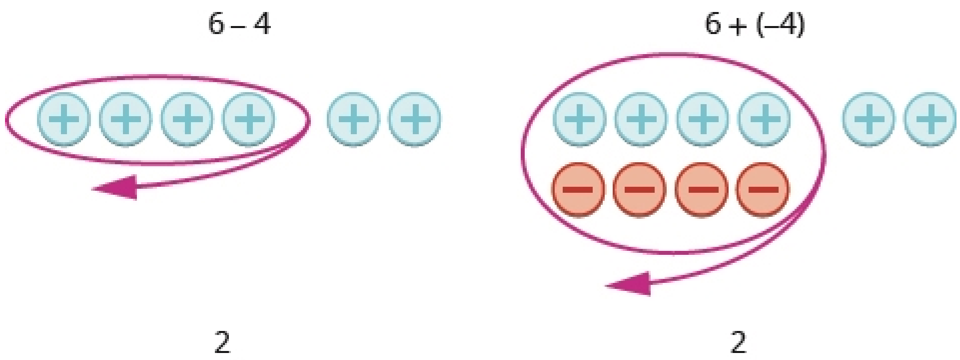
While we may not always use the counters, especially when we work with large numbers, practicing with them first gave us a concrete way to apply the concept, so that we can visualize and remember how to do the subtraction without the counters.

Have you noticed that subtraction of signed numbers can be done by adding the opposite? You will often see the idea, the Subtraction Property, written as follows:

Note:
Subtraction Property
Equation:

$$a-b=a+(-b)$$

Look at these two examples.



We see that $6-4$ gives the same answer as $6+(-4)$.

Of course, when we have a subtraction problem that has only positive numbers, like the first example, we just do the subtraction. We already knew how to subtract $6-4$ long ago. But knowing that $6-4$ gives the same answer as$6+(-4)$ helps when we are adding negative numbers.

Example:
Exercise:

Problem: Simplify:

ⓐ $13-8$ and $13+(-8)$
ⓑ $-17-9$ and $-17+(-9)$

Solution:
Solution

ⓐ	
	$13-8$ and

	$13+(-8)$
Subtract to simplify.	$13-8=5$
Add to simplify.	$13+(-8)=5$
Subtracting 8 from 13 is the same as adding −8 to 13.	

ⓑ	
	$-17-9$ and $-17+(-9)$
Subtract to simplify.	$-17-9=-26$
Add to simplify.	$-17+(-9)=-26$
Subtracting 9 from −17 is the same as adding −9 to −17.	

Note:
Exercise:

Problem: Simplify each expression:

ⓐ $21-13$ and $21+(-13)$
ⓑ $-11-7$ and $-11+(-7)$

Solution:

ⓐ 8, 8
ⓑ −18, −18

Note:
Exercise:

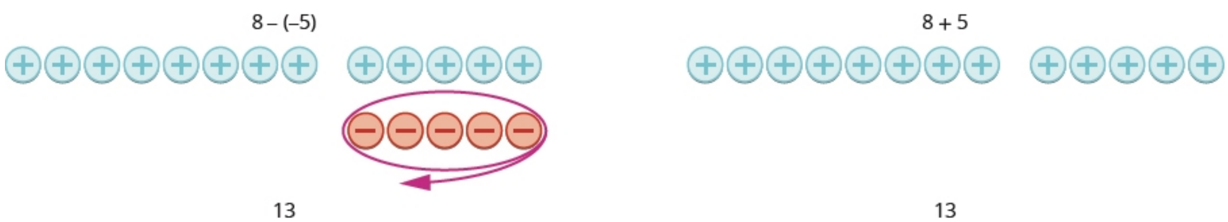
Problem: Simplify each expression:

ⓐ $15 - 7$ and $15 + (-7)$
ⓑ $-14 - 8$ and $-14 + (-8)$

Solution:

ⓐ 8, 8
ⓑ −22, −22

Now look what happens when we subtract a negative.



We see that $8 - (-5)$ gives the same result as $8 + 5$. Subtracting a negative number is like adding a positive.

Example:
Exercise:

Problem: Simplify:

ⓐ $9 - (-15)$ and $9 + 15$
ⓑ $-7 - (-4)$ and $-7 + 4$

Solution:
Solution

ⓐ	
	$9 - (-15)$ and $9 + 15$
Subtract to simplify.	$9 - (-15) = -24$
Add to simplify.	$9 + 15 = 24$
Subtracting −15 from 9 is the same as adding 15 to 9.	

ⓑ	
	$-7 - (-4)$ and

	$-7+4$
Subtract to simplify.	$-7-(-4)=-3$
Add to simplify.	$-7+4=-3$
Subtracting −4 from −7 is the same as adding 4 to −7	

Note:
Exercise:

Problem: Simplify each expression:

ⓐ $6-(-13)$ and $6+13$
ⓑ $-5-(-1)$ and $-5+1$

Solution:

ⓐ 19, 19
ⓑ −4, −4

Note:
Exercise:

Problem: Simplify each expression:

ⓐ $4-(-19)$ and $4+19$
ⓐ $-4-(-7)$ and $-4+7$

Solution:

ⓐ 23, 23
ⓑ 3, 3

Look again at the results of [link] - [link].

$5-3$	$-5-(-3)$
2	-2
2 positives	2 negatives
When there would be enough counters of the color to take away, subtract.	
$-5-3$	$5-(-3)$
-8	8
5 negatives, want to subtract 3 positives	5 positives, want to subtract 3 negatives
need neutral pairs	need neutral pairs
When there would not be enough of the counters to take away, add neutral pairs.	

Subtraction of Integers

Example:
Exercise:

Problem: Simplify: $-74-(-58)$.

Solution:
Solution

We are taking 58 negatives away from 74 negatives.	$-74-(-58)$
Subtract.	-16

Note:
Exercise:

Problem: Simplify the expression:

$-67-(-38)$

Solution:

-29

Note:
Exercise:

Problem: Simplify the expression:

$-83-(-57)$

Solution:

-26

Example:
Exercise:

Problem: Simplify: $7-(-4-3)-9$.

Solution:
Solution

We use the order of operations to simplify this expression, performing operations inside the parentheses first. Then we subtract from left to right.

Simplify inside the parentheses first.	$7-(-4-3)-9$
Subtract from left to right.	$7-(-7)-9$

Subtract.	14 – 9
	5

Note:
Exercise:

Problem: Simplify the expression:

$8 - (-3 - 1) - 9$

Solution:

3

Note:
Exercise:

Problem: Simplify the expression:

$12 - (-9 - 6) - 14$

Solution:

13

Example:
Exercise:

Problem: Simplify: $3 \cdot 7 - 4 \cdot 7 - 5 \cdot 8$.

Solution:
Solution

We use the order of operations to simplify this expression. First we multiply, and then subtract from left to right.

Multiply first.	$3 \cdot 7 - 4 \cdot 7 - 5 \cdot 8$
Subtract from left to right.	$21 - 28 - 40$
Subtract.	$-7 - 40$
	-47

Note:
Exercise:

Problem: Simplify the expression:

$6 \cdot 2 - 9 \cdot 1 - 8 \cdot 9.$

Solution:

−69

Note:
Exercise:

Problem: Simplify the expression:

$2 \cdot 5 - 3 \cdot 7 - 4 \cdot 9$

Solution:

−47

Properties of Subtraction

The Identity Property of Subtraction

$a - 0 = a$

Why is this true? We already know that it is true for whole numbers. Is it also true for integers.
In other words, does the property still hold for the expanded set of numbers that also includes negatives?

For example, does -3 - 0 = -3? Yes it does. Why?
If I have any number and take away nothing, then the number remains the same.

Another property is that any number minus itself equals 0. If we have any number of counters and take exactly that number away, then exactly nothing is left. Since subtraction can be thought of as adding the opposite, this is the same as the Addition Property of Opposites.
$a - a = 0$

Other Properties of Subtraction
Addition also has a commutative and an associative property. But since subtraction does not have these properties even for just whole numbers then it cannot have the properties for all integers since that includes the whole numbers. Therefore subtraction is not commutative or associative.

Adding Instead of Subtracting
The Subtraction Property allows us to replace any subtraction with adding the opposite. One reason we might want to do this is that it would produce an equivalent expression that has terms that can be rearranged since the resulting expression is both commutative and associative.

Consider: 937 + 165 - 937.
If we do this calculation is a straight forward fashion we get:
937 + 165 = 1,102.
1,102 - 937 = 165.
Done by hand these calculations take a little work.

Here is the same calculation done differently:
changing the subtraction to adding the opposite 937 + 165 + (-937)
Since addition is commutative 165 + 937 + (-937)
Since addition is associative 165 + (937 + (-937))
Since opposites add to zero 165 + 0
The Identity Property of Addition 165

While a lot was written for this problem, with a little practice you would immediately see that the answer was 165.
What this shows is that on a purely logical level, having negative numbers removes the need to ever subtract.
As a practical matter, we can still think in terms of subtraction but mathematically subtraction is unnecessary once you have negative numbers.

Centuries ago many mathematicians were reluctant to accept the idea of negative numbers. But eventually mathematicians saw that using negative numbers ultimately made doing mathematics easier rather than harder.

Evaluate Variable Expressions with Integers

Now we'll practice evaluating expressions that involve subtracting negative numbers as well as positive numbers.

Example:
Exercise:

Problem: Evaluate $x - 4$ when

ⓐ $x = 3$
ⓑ $x = -6.$

Solution:
Solution

ⓐ To evaluate $x - 4$ when $x = 3$, substitute 3 for x in the expression.

	$x - 4$
Substitute 3 for x.	$3 - 4$

Subtract.	-1

ⓑ To evaluate $x - 4$ when $x = -6$, substitute -6 for x in the expression.

	$x - 4$
Substitute -6 for x.	$-6 - 4$
Subtract.	-10

Note:
Exercise:

Problem: Evaluate each expression:

$y - 7$ when

ⓐ $y = 5$
ⓑ $y = -8$

Solution:

ⓐ -2
ⓑ -15

Note:
Exercise:

Problem: Evaluate each expression:

$m - 3$ when

ⓐ $m = 1$
ⓑ $m = -4$

Solution:

ⓐ -2
ⓑ -7

Example:
Exercise:

Problem: Evaluate $20 - z$ when

ⓐ $z = 12$
ⓑ $z = -12$

Solution:
Solution

ⓐ To evaluate $20 - z$ when $z = 12$, substitute 12 for z in the expression.

	$20 - z$
Substitute 12 for z.	$20 - 12$
Subtract.	8

ⓑ To evaluate
$20 - z$ when $z = -12,$ substitute -12 for z in the expression.

	$20 - z$
Substitute −12 for z.	$20 - (-12)$

Subtract.	32

Note:
Exercise:

Problem: Evaluate each expression:

$17 - k$ when

ⓐ $k = 19$
ⓑ $k = -19$

Solution:

ⓐ −2
ⓑ 36

Note:
Exercise:

Problem: Evaluate each expression:

$-5 - b$ when

ⓐ $b = 14$
ⓑ $b = -14$

Solution:

ⓐ −19
ⓑ 9

Translate Word Phrases to Algebraic Expressions

When we first introduced the operation symbols, we saw that the expression $a - b$ may be read in several ways as shown below.

a* – *b
a minus *b*
the difference of *a* and *b*
subtract *b* from *a*
b subtracted from *a*
b less than *a*

Be careful to get a and b in the right order!

Example:
Exercise:

Translate and then simplify:
Problem: ⓐ the difference of 13 and -21

ⓑ subtract 24 from -19

Solution:
Solution

ⓐ A *difference* means subtraction. Subtract the numbers in the order they are given.

	the difference of 13 and -21
Translate.	$13 - (-21)$
Simplify.	34

ⓑ *Subtract* means to take 24 away from -19.

	subtract 24 from -19
Translate.	$-19 - 24$
Simplify.	-43

Note:
Exercise:

Problem: Translate and simplify:

ⓐ the difference of 14 and -23
ⓑ subtract 21 from -17

Solution:

ⓐ $-14 - (-23) = 37$
ⓑ $-17 - 21 = -38$

Note:
Exercise:

Problem: Translate and simplify:

ⓐ the difference of 11 and -19
ⓑ subtract 18 from -11

Solution:

ⓐ $11 - (-19) = 30$
ⓑ $-11 - 18 = -29$

Subtract Integers in Applications

It's hard to find something if we don't know what we're looking for or what to call it. So when we solve an application problem, we first need to determine what we are asked to find. Then we can write a phrase that gives the information to find it. We'll translate the phrase into an expression and then simplify the expression to get the answer. Finally, we summarize the answer in a sentence to make sure it makes sense.

Note:
Solve Application Problems.

Identify what you are asked to find.
Write a phrase that gives the information to find it.
Translate the phrase to an expression.
Simplify the expression.
Answer the question with a complete sentence.

Example:
Exercise:

Problem:

The temperature in Urbana, Illinois one morning was 11 degrees Fahrenheit. By mid-afternoon, the temperature had dropped to -9 degrees Fahrenheit. What was the difference between the morning and afternoon temperatures?

Solution:
Solution

Step 1. Identify what we are asked to find.	the difference between the morning and afternoon temperatures
Step 2. Write a phrase that gives the information to find it.	the difference of 11 and -9
Step 3. Translate the phrase to an expression. The word *difference* indicates subtraction.	$11-(-9)$
Step 4. Simplify the expression.	20
Step 5. Write a complete sentence that answers the question.	The difference in temperature was 20 degrees Fahrenheit.

Note:
Exercise:

Problem:

The temperature in Anchorage, Alaska one morning was $15\text{ degrees Fahrenheit}$. By mid-afternoon the temperature had dropped to 30 degrees below zero. What was the difference between the morning and afternoon temperatures?

Solution:

45 degrees Fahrenheit

Note:
Exercise:

Problem:

The temperature in Denver was -6 degrees Fahrenheit at lunchtime. By sunset the temperature had dropped to $-15 \text{ degree Fahrenheit}.$ What was the difference between the lunchtime and sunset temperatures?

Solution:

9 degrees Fahrenheit

Geography provides another application of negative numbers with the elevations of places below sea level.

Example:
Exercise:

Problem:

Dinesh hiked from Mt. Whitney, the highest point in California, to Death Valley, the lowest point. The elevation of Mt. Whitney is $14,497$ feet above sea level and the elevation of Death Valley is 282 feet below sea level. What is the difference in elevation between Mt. Whitney and Death Valley?

Solution:
Solution

Step 1. What are we asked to find?	The difference in elevation between Mt. Whitney and Death Valley
Step 2. Write a phrase.	elevation of Mt. Whitney−elevation of Death Valley
Step 3. Translate.	$14{,}497-(-282)$
Step 4. Simplify.	$14{,}779$
Step 5. Write a complete sentence that answers the question.	The difference in elevation is $14{,}779$ feet.

Note:
Exercise:

Problem:

One day, John hiked to the $10{,}023$ foot summit of Haleakala volcano in Hawaii. The next day, while scuba diving, he dove to a cave 80 feet below sea level. What is the difference between the elevation of the summit of Haleakala and the depth of the cave?

Solution:

10,103 feet

Note:

Exercise:

Problem:

The submarine Nautilus is at 340 feet below the surface of the water and the submarine Explorer is 573 feet below the surface of the water. What is the difference in the position of the Nautilus and the Explorer?

Solution:

233 feet

Managing your money can involve both positive and negative numbers. You might have overdraft protection on your checking account. This means the bank lets you write checks for more money than you have in your account (as long as they know they can get it back from you!)

Example:
Exercise:

Problem:

Leslie has $\$25$ in her checking account and she writes a check for $\$8$.

ⓐ What is the balance after she writes the check?
ⓑ She writes a second check for $\$20$. What is the new balance after this check?
ⓒ Leslie's friend told her that she had lost a check for $\$10$ that Leslie had given her with her birthday card. What is the balance in Leslie's checking account now?

Solution:
Solution

ⓐ	
What are we asked to find?	The balance of the account
Write a phrase.	\$25 minus \$8
Translate	\$25 − \$8
Simplify.	\$17
Write a sentence answer.	The balance is \$17.

ⓑ	
What are we asked to find?	The new balance
Write a phrase.	\$17 minus \$20
Translate	\$17 − \$20
Simplify.	−\$3
Write a sentence answer.	She is overdrawn by \$3.

ⓒ	
What are we asked to find?	The new balance
Write a phrase.	$10 more than −$3
Translate	−$3 + $10
Simplify.	$7
Write a sentence answer.	The balance is now $7.

Note:
Exercise:

Problem:

Araceli has $75 in her checking account and writes a check for $27.

ⓐ What is the balance after she writes the check?
ⓑ She writes a second check for $50. What is the new balance?
ⓒ The check for $20 that she sent a charity was never cashed. What is the balance in Araceli's checking account now?

Solution:

ⓐ $48
ⓑ −$2
ⓒ $18

Note:
Exercise:

Problem:

Genevieve's bank account was overdrawn and the balance is $-\$78$.

ⓐ She deposits a check for $\$24$ that she earned babysitting. What is the new balance?
ⓑ She deposits another check for $\$49$. Is she out of debt yet? What is her new balance?

Solution:

ⓐ −$54
ⓑ No, −$5

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Adding and Subtracting Integers
- Subtracting Integers with Color Counters
- Subtracting Integers Basics
- Subtracting Integers
- Integer Application

Key Concepts

- **Subtraction of Integers**

$5 - 3$	$-5 - (-3)$
2	-2
2 positives	2 negatives
When there would be enough counters of the color to take away, subtract.	
$-5 - 3$	$5 - (-3)$
-8	8
5 negatives, want to subtract 3 positives	5 positives, want to subtract 3 negatives
need neutral pairs	need neutral pairs
When there would not be enough of the counters to take away, add neutral pairs.	

- **Subtraction Property**
    - $a - b = a + (-b)$
    - $a - (-\mathrm{b}) = a + b$
- **Solve Application Problems**
    - Step 1. Identify what you are asked to find.
    - Step 2. Write a phrase that gives the information to find it.

- Step 3. Translate the phrase to an expression.
- Step 4. Simplify the expression.
- Step 5. Answer the question with a complete sentence.

Practice Makes Perfect

Model Subtraction of Integers

In the following exercises, model each expression and simplify.

Exercise:

Problem: $8 - 2$

Solution:

6

Exercise:

Problem: $9 - 3$

Exercise:

Problem: $-5 - (-1)$

Solution:

-4

Exercise:

Problem: $-6-(-4)$

Exercise:

Problem: $-5-4$

Solution:

-9

Exercise:

Problem: $-7-2$

Exercise:

Problem: $8-(-4)$

Solution:

12

Exercise:

Problem: $7 - (-3)$

Simplify Expressions with Integers

In the following exercises, simplify each expression.

Exercise:

Problem:

ⓐ $15 - 6$
ⓑ $15 + (-6)$

Solution:

ⓐ 9
ⓑ 9

Exercise:

Problem:

ⓐ $12 - 9$
ⓑ $12 + (-9)$

Exercise:

Problem:

ⓐ $44 - 28$
ⓑ $44 + (-28)$

Solution:

ⓐ 16
ⓑ 16

Exercise:

Problem:

ⓐ $35 - 16$
ⓑ $35 + (-16)$

Exercise:

Problem:

ⓐ $8 - (-9)$
ⓑ $8 + 9$

Solution:

ⓐ 17
ⓑ 17

Exercise:

Problem:

ⓐ $4 - (-4)$
ⓑ $4 + 4$

Exercise:

Problem:

ⓐ $27 - (-18)$
ⓑ $27 + 18$

Solution:

ⓐ 45
ⓑ 45

Exercise:

Problem:

ⓐ $46 - (-37)$
ⓑ $46 + 37$

In the following exercises, simplify each expression.

Exercise:

Problem: $15 - (-12)$

Solution:

27

Exercise:

Problem: $14 - (-11)$

Exercise:

Problem: $10 - (-19)$

Solution:

29

Exercise:

Problem: $11 - (-18)$

Exercise:

Problem: $48 - 87$

Solution:

-39

Exercise:

Problem: $45 - 69$

Exercise:

Problem: $31 - 79$

Solution:

-48

Exercise:

Problem: $39 - 81$

Exercise:

Problem: $-31 - 11$

Solution:

-42

Exercise:

Problem: $-32 - 18$

Exercise:

Problem: $-17 - 42$

Solution:

-59

Exercise:

Problem: $-19 - 46$

Exercise:

Problem: $-103 - (-52)$

Solution:

-51

Exercise:

Problem: $-105 - (-68)$

Exercise:

Problem: $-45 - (-54)$

Solution:

9

Exercise:

Problem: $-58 - (-67)$

Exercise:

Problem: $8 - 3 - 7$

Solution:

-2

Exercise:

Problem: $9-6-5$

Exercise:

Problem: $-5-4+7$

Solution:

-2

Exercise:

Problem: $-3-8+4$

Exercise:

Problem: $-14-(-27)+9$

Solution:

22

Exercise:

Problem: $-15-(-28)+5$

Exercise:

Problem: $71+(-10)-8$

Solution:

53

Exercise:

Problem: $64 + (-17) - 9$

Exercise:

Problem: $-16 - (-4 + 1) - 7$

Solution:

-20

Exercise:

Problem: $-15 - (-6 + 4) - 3$

Exercise:

Problem: $(2 - 7) - (3 - 8)$

Solution:

0

Exercise:

Problem: $(1 - 8) - (2 - 9)$

Exercise:

Problem: $-(6 - 8) - (2 - 4)$

Solution:

4

Exercise:

Problem: $-(4 - 5) - (7 - 8)$

Exercise:

Problem: $25 - [10 - (3 - 12)]$

Solution:

6

Exercise:

Problem: $32 - [5 - (15 - 20)]$

Exercise:

Problem: $6 \cdot 3 - 4 \cdot 3 - 7 \cdot 2$

Solution:

-8

Exercise:

Problem: $5 \cdot 7 - 8 \cdot 2 - 4 \cdot 9$

Exercise:

Problem: $5^2 - 6^2$

Solution:

-11

Exercise:

Problem: $6^2 - 7^2$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression for the given values.

Exercise:

Problem: $x - 6$ when

ⓐ $x = 3$
ⓑ $x = -3$

Solution:

ⓐ -3
ⓑ -9

Exercise:

Problem: $x - 4$ when

ⓐ $x = 5$
ⓑ $x = -5$

Exercise:

Problem: $5 - y$ when

ⓐ $y = 2$
ⓑ $y = -2$

Solution:

ⓐ 3
ⓑ 7

Exercise:

Problem: $8 - y$ when

ⓐ $y = 3$
ⓑ $y = -3$

Exercise:

Problem: $4x^2 - 15x + 1$ when $x = 3$

Solution:

−8

Exercise:

Problem: $5x^2 - 14x + 7$ when $x = 2$

Exercise:

Problem: $-12 - 5x^2$ when $x = 6$

Solution:

−192

Exercise:

Problem: $-19 - 4x^2$ when $x = 5$

Translate Word Phrases to Algebraic Expressions

In the following exercises, translate each phrase into an algebraic expression and then simplify.

Exercise:

Problem:

ⓐ The difference of 3 and -10
ⓑ Subtract -20 from 45

Solution:

ⓐ −3 − (−10) = 13
ⓑ 45 − (−20) = 65

Exercise:

Problem:

ⓐ The difference of 8 and -12
ⓑ Subtract -13 from 50

Exercise:

Problem:

ⓐ The difference of -6 and 9
ⓑ Subtract -12 from -16

Solution:

ⓐ −6 − 9 = −15
ⓑ −16 − (−12) = −4

Exercise:

Problem:

ⓐ The difference of -8 and 9
ⓑ Subtract -15 from -19

Exercise:

Problem:

ⓐ 8 less than -17
ⓑ -24 minus 37

Solution:

ⓐ $-17 - 8 = -25$
ⓑ $-24 - 37 = -61$

Exercise:

Problem:

ⓐ 5 less than -14
ⓑ -13 minus 42

Exercise:

Problem:

ⓐ 21 less than 6
ⓑ 31 subtracted from -19

Solution:

ⓐ $6 - 21 = -15$
ⓑ $-19 - 31 = -50$

Exercise:

Problem:

ⓐ 34 less than 7
ⓑ 29 subtracted from -50

Subtract Integers in Applications

In the following exercises, solve the following applications.

Exercise:

Problem:

Temperature One morning, the temperature in Urbana, Illinois, was $28^\circ\ \text{Fahrenheit}$. By evening, the temperature had dropped $38^\circ\ \text{Fahrenheit}$. What was the temperature that evening?

Solution:

−10°

Exercise:

Problem:

Temperature On Thursday, the temperature in Spincich Lake, Michigan, was $22^\circ\ \text{Fahrenheit}$. By Friday, the temperature had dropped $35^\circ\ \text{Fahrenheit}$. What was the temperature on Friday?

Exercise:

Problem:

Temperature On January 15, the high temperature in Anaheim, California, was $84^\circ\ \text{Fahrenheit}$. That same day, the high temperature in Embarrass, Minnesota was $-12^\circ\ \text{Fahrenheit}$. What was the difference between the temperature in Anaheim and the temperature in Embarrass?

Solution:

96°

Exercise:

Problem:

Temperature On January 21, the high temperature in Palm Springs, California, was 89°, and the high temperature in Whitefield, New Hampshire was -31°. What was the difference between the temperature in Palm Springs and the temperature in Whitefield?

Exercise:

Problem:

Football At the first down, the Warriors football team had the ball on their 30-yard line. On the next three downs, they gained 2 yards, lost 7 yards, and lost 4 yards. What was the yard line at the end of the third down?

Solution:

21-yard line

Exercise:

Problem:

Football At the first down, the Barons football team had the ball on their 20-yard line. On the next three downs, they lost 8 yards, gained 5 yards, and lost 6 yards. What was the yard line at the end of the third down?

Exercise:

Problem:

Checking Account John has $\$148$ in his checking account. He writes a check for $\$83$. What is the new balance in his checking account?

Solution:

$65

Exercise:

Problem:

Checking Account Ellie has $426 in her checking account. She writes a check for $152. What is the new balance in her checking account?

Exercise:

Problem:

Checking Account Gina has $210 in her checking account. She writes a check for $250. What is the new balance in her checking account?

Solution:

$-\$40$

Exercise:

Problem:

Checking Account Frank has $94 in his checking account. He writes a check for $110. What is the new balance in his checking account?

Exercise:

Problem:

Checking Account Bill has a balance of $-\$14$ in his checking account. He deposits $40 to the account. What is the new balance?

Solution:

$26

Exercise:

Problem:

Checking Account Patty has a balance of $-\$23$ in her checking account. She deposits $80 to the account. What is the new balance?

Everyday Math

Exercise:

Problem:

Camping Rene is on an Alpine hike. The temperature is -7°. Rene's sleeping bag is rated "comfortable to -20°". How much can the temperature change before it is too cold for Rene's sleeping bag?

Solution:

13°

Exercise:

Problem:

Scuba Diving Shelly's scuba watch is guaranteed to be watertight to -100 feet. She is diving at -45 feet on the face of an underwater canyon. By how many feet can she change her depth before her watch is no longer guaranteed?

Writing Exercises

Exercise:

Problem: Explain why the difference of 9 and -6 is 15.

Solution:

Sample answer: On a number line, 9 is 15 units away from −6.

Exercise:

Problem:

Why is the result of subtracting $3-(-4)$ the same as the result of adding $3+4$?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
model subtraction of integers.			
simplify expressions with integers.			
evaluate variable expressions with integers.			
translate word phrases to algebraic expressions.			
subtract integers in applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Multiply and Divide Integers
By the end of this section, you will be able to:

- Multiply integers
- Divide integers
- Simplify expressions with integers
- Evaluate variable expressions with integers
- Translate word phrases to algebraic expressions

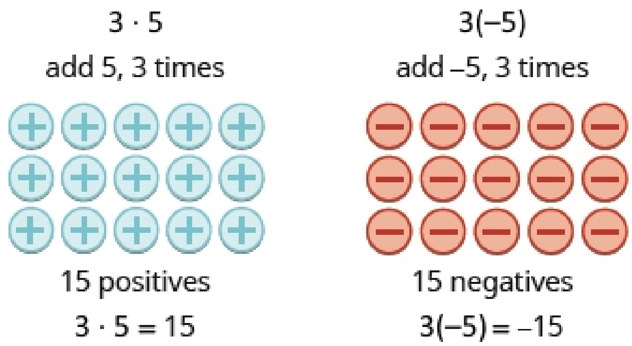
Note:
Before you get started, take this readiness quiz.

1. Translate the quotient of 20 and 13 into an algebraic expression.
   If you missed this problem, review [link].
2. Add: $-5+(-5)+(-5)$.
   If you missed this problem, review [link].
3. Evaluate $n+4$ when $n=-7$.
   If you missed this problem, review [link].

Multiply Integers

Since multiplication can be thought of as repeated addition, our counter model can show multiplication of integers. Let's look at this concrete model to see what patterns we notice. We will use the same examples that we used for addition and subtraction.

We remember that $a\cdot b$ means add a addends of b. Here, we are using the model shown in [link] just to help us discover the pattern.

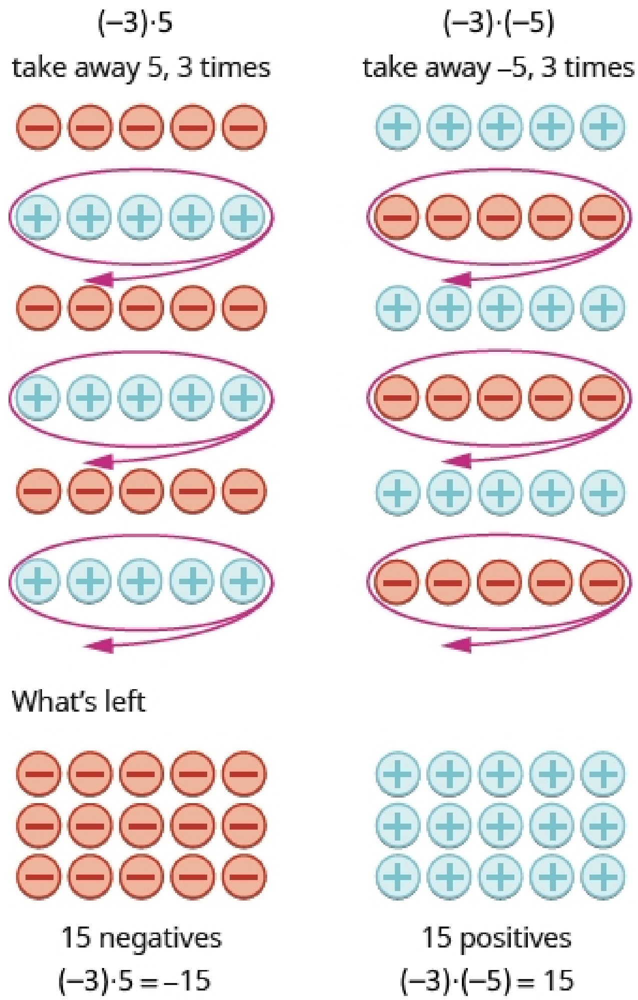


A positive integer times a positive integer is the same as for whole numbers. Therefore a positive times a positive results in a positive product. A positive integer times a negative integer is easy to understand as repeatedly adding the same negative addend. Since a negative plus a negative results in a negative sum, it follows that a positive times a negative must be negative.

What about a negative integer times a positive integer? Here our intuition is probably less certain because it does not immediately make sense how to do anything a negative number of times. One way this can make sense is to recognize that negative and positive integers are opposites of each other, and that addition and subtraction are opposite operations or inverses of each other because adding and subtracting the same number results in no net change. Consequently, if multiplying by a positive integer is equivalent to repeated addition, multiplying by a negative integer is equivalent to repeated subtraction.

Consider what it means to multiply -3 times 5. One meaning is subtract 5, 3 times. Starting with nothing, we cannot do the subtraction. Therefore, we start by adding neutral pairs as shown on the left side of [link]. This is one way of understanding why a negative integer times a positive integer results in a negative product.

What about a negative integer times a negative integer? The right hand side of [link] shows that it is positive. Notice that we are taking away the negative counters, leaving the positive counters. This is consistent with when we subtracted a negative value and saw that was the same as adding the positive value that was its opposite. Doing this repeatedly gives the same result as multiplying the opposites of the two negative values. Therefore a negative integer times a negative integer must result in a positive product.



In both cases, we started with $\mathbf{15}$ neutral pairs. In the case on the left, we took away $\mathbf{5}$, $\mathbf{3}$ times and the result was $-\mathbf{15}$. To multiply $(-3)\,(-5)$, we took away $-\mathbf{5}$, $\mathbf{3}$ times and the result was $\mathbf{15}$. So we found that

Equation:

$$\begin{array}{ll} 3 \cdot 5 = 15 & 3(-5) = -15 \\ -3(5) = -15 & (-3)(-5) = 15 \end{array}$$

Another way of appreciating why a negative integer times a positive integer results in a negative product relies on the commutative property. When the natural numbers are extended to become the integers, we want all of the properties that were true for natural numbers to still be true when we think of them as integers. This includes the commutative property of multiplication: a x b = b x a. If we require this property to be true for all integers, including the

negative ones, then -3 x 5 = 5 x (-3), and a negative times a positive must be negative since we've already seen that a positive times a negative is negative.

Just as we needed a negative integer times a positive integer to result in a negative product in order for the commutative property to continue to be true, we need a negative integer times a negative integer to have a positive product for the distributive property to hold. Consider the following equations and consider a reason why each one is true:

- 5 + (-5) = 0
  Reason: any number plus its opposite = 0
- (-3) x (5 + (-5)) = 0
  Reason: any number times 0 = 0
- (-3) x 5 + (-3) x (-5) = 0
  Reason: the distributive property of multiplication over addition
- -15 + (-3) x (-5) = 0
  Reason: multiplication fact and a negative times a positive is negative
- -15 and (-3) x (-5) are opposites
  Reason: they add to zero
- -(-15) = 15
  Reason: the opposite of an opposite is the number itself
- (-3) x (-5) = 15
  Reason: (-3) x (-5) is also the opposite of -15

This is another way of understanding why a negative number times a negative number results in a positive product.

Notice that for multiplication of two signed numbers, when the signs are the same, the product is positive, and when the signs are different, the product is negative.

Note:
Multiplication of Signed Numbers
The sign of the product of two numbers depends on their signs.

Same signs	Product
•Two positives •Two negatives	Positive Positive

Different signs	Product
•Positive • negative •Negative • positive	Negative Negative

Example:
Exercise:

Problem: Multiply each of the following:

ⓐ $-9 \cdot 3$
ⓑ $-2(-5)$
ⓒ $4(-8)$
ⓓ $7 \cdot 6$

Solution:
Solution

ⓐ	

	$-9 \cdot 3$
Multiply, noting that the signs are different and so the product is negative.	-27

ⓑ	
	$-2(-5)$
Multiply, noting that the signs are the same and so the product is positive.	10

ⓒ	
	$4(-8)$
Multiply, noting that the signs are different and so the product is negative.	-32

ⓓ	
	$7 \cdot 6$

The signs are the same, so the product is positive.	42

Note:
Exercise:

Problem: Multiply:

ⓐ $-6 \cdot 8$
ⓑ $-4(-7)$
ⓒ $9(-7)$
ⓓ $5 \cdot 12$

Solution:

ⓐ −48
ⓑ 28
ⓒ −63
ⓓ 60

Note:
Exercise:

Problem: Multiply:

- ⓐ $-8 \cdot 7$
- ⓑ $-6(-9)$
- ⓒ $7(-4)$
- ⓓ $3 \cdot 13$

Solution:

ⓐ −56
ⓑ 54
ⓒ −28
ⓓ 39

When we multiply a number by 1, the result is the same number. This is the Identity Property of Multiplication. What happens when we multiply a number by-1? Let's multiply a positive number and then a negative number by -1 to see what we get.

Equation:

$$\begin{array}{cc} -1 \cdot 4 & -1(-3) \\ -4 & 3 \\ -4 \text{ is the opposite of } \mathbf{4} & \mathbf{3} \text{ is the opposite of } -3 \end{array}$$

Each time we multiply a number by -1, we get its opposite.

Note:
Multiplication by -1
Multiplying a number by -1 gives its opposite.
Equation:

$$-1a = -a$$

Example:
Exercise:

Problem: Multiply each of the following:

ⓐ $-1 \cdot 7$
ⓑ $-1(-11)$

Solution:
Solution

ⓐ	
The signs are different, so the product will be negative.	$-1 \cdot 7$
Notice that −7 is the opposite of 7.	-7

ⓑ	
The signs are the same, so the product will be positive.	$-1(-11)$
Notice that 11 is the opposite of −11.	11

Note:
Exercise:

Problem: Multiply.

ⓐ $-1 \cdot 9$
ⓑ $-1 \cdot (-17)$

Solution:

ⓐ -9
ⓑ 17

Note:
Exercise:

Problem: Multiply.

ⓐ $-1 \cdot 8$
ⓑ $-1 \cdot (-16)$

Solution:

ⓐ -8
ⓑ 16

Properties of Multiplication
All of the properties of multiplication for whole numbers hold for integers. That includes:
Identity Property of Multiplication: a x 1 = a
Zero Property of Multiplication: a x 0 = 0
Commutative Property of Multiplication: a x b = b x a
Associative Property of Multiplication: (a x b) x c = a x (b x c)
Distributive Property of Multiplication over Addition: a x (b + c) = a x b + a x d

Divide Integers

Division is the inverse operation of multiplication. So, $15 \div 3 = 5$ because $3 \cdot 5 = 15$ In words, this expression says that $\mathbf{15}$ can be divided into $\mathbf{3}$ groups of

$\mathbf{5}$ each because adding five three times gives $\mathbf{15}$. If we look at some examples of multiplying integers, we can discover the rules for dividing integers.
Equation:

$3 \cdot 5 = 15 \text{ so } 15 \div 3 = 5$

$3(-5) = -15 \text{ so } -15 \div 3 = -5$

$(-3)(-5) = 15 \text{ so } 15 \div (-3) = -5$

$-3(5) = -15 \text{ so } -15 \div (-3) = 5$

Division of signed numbers follows the same rules as multiplication. When the signs of the divisor and dividend are the same, the quotient is positive, and when the signs are different, the quotient is negative.

Note:
Division of Signed Numbers
The sign of the quotient of two numbers depends on the signs of the divisor and dividend.

Same signs	**Quotient**
•Two positives •Two negatives	Positive Positive

Different signs	**Quotient**
•Positive & negative •Negative & positive	Negative Negative

Notice that this is exactly the same summary we had for multiplication. Also, you can always check the answer to a division problem by multiplying.

Example:
Exercise:

Problem: Divide each of the following:

ⓐ $-27 \div 3$
ⓑ $-100 \div (-4)$

Solution:
Solution

ⓐ	
	$-27 \div 3$
Divide, noting that the signs are different and so the quotient is negative.	-9

ⓑ	
	$-100 \div (-4)$
Divide, noting that the signs are the same and so the quotient is positive.	25

Note:
Exercise:

Problem: Divide:

ⓐ $-42 \div 6$
ⓑ $-117 \div (-3)$

Solution:

ⓐ −7
ⓑ 39

Note:
Exercise:

Problem: Divide:

ⓐ $-63 \div 7$
ⓑ $-115 \div (-5)$

Solution:

ⓐ −9
ⓑ 23

Just as we saw with multiplication, when we divide a number by 1, the result is the same number. What happens when we divide a number by -1? Let's divide a positive number and then a negative number by -1 to see what we get.
Equation:

$8 \div (-1)$	$-9 \div (-1)$
-8	9
-8 is the opposite of 8	9 is the opposite of -9

When we divide a number by, -1 we get its opposite.

Note:
Division by -1
Dividing a number by -1 gives its opposite.
Equation:

$$a \div (-1) = -a$$

Example:
Exercise:

Problem: Divide each of the following:

ⓐ $16 \div (-1)$
ⓑ $-20 \div (-1)$

Solution:
Solution

ⓐ	
	$16 \div (-1)$
The dividend, 16, is being divided by –1.	–16
Dividing a number by –1 gives its opposite.	
Notice that the signs were different, so the result was negative.	

ⓑ	
	$-20 \div (-1)$
The dividend, –20, is being divided by –1.	20
Dividing a number by –1 gives its opposite.	

Notice that the signs were the same, so the quotient was positive.

Note:
Exercise:

Problem: Divide:

ⓐ $6 \div (-1)$
ⓑ $-36 \div (-1)$

Solution:

ⓐ -6
ⓑ 36

Note:
Exercise:

Problem: Divide:

ⓐ $28 \div (-1)$
ⓑ $-52 \div (-1)$

Solution:

ⓐ -28
ⓑ 52

Properties of Division

Recall that with whole numbers division has some properties that multiplication has but other properties fail. For example, division is neither commutative or associative with whole numbers and since integers includes whole numbers, division with integers is neither commutative or associative.

- Division by zero is still undefined
- 0 divided by any number other than 0 is 0
- Any number divided by 1 is the same number
- Any number divided by itself other than 0 is 1
- The distributive property of division over addition: (a+b)/c = a/c + b/c
  This property can involve fractions so it will be discussed further in the chapter on fractions

Simplify Expressions with Integers

Now we’ll simplify expressions that use all four operations–addition, subtraction, multiplication, and division–with integers. Remember to follow the order of operations.

Example:
Exercise:

Problem: Simplify: $7(-2) + 4(-7) - 6$.

Solution:
Solution

We use the order of operations. Multiply first and then add and subtract from left to right.

	$7(-2) + 4(-7)-6$
Multiply first.	$-14 + (-28)-6$
Add.	$-42 - 6$
Subtract.	-48

Note:
Exercise:

Problem: Simplify:

$8(-3) + 5(-7)-4$

Solution:

-63

Note:
Exercise:

Problem: Simplify:

$9(-3) + 7(-8) - 1$

Solution:

-84

Example:
Exercise:

Problem: Simplify:

ⓐ $(-2)^4$
ⓑ -2^4

Solution:
Solution

The exponent tells how many factors there are of the base.

ⓐ The exponent is 4 and the base is -2. We raise -2 to the fourth power.

	$(-2)^4$
Write in expanded form.	$(-2)(-2)(-2)(-2)$
Multiply.	$4(-2)(-2)$
Multiply.	$-8(-2)$
Multiply.	16

ⓑ The exponent is 4 and the base is 2. We raise 2 to the fourth power and then take the opposite.

	-2^4
Write in expanded form.	$-(2 \cdot 2 \cdot 2 \cdot 2)$
Multiply.	$-(4 \cdot 2 \cdot 2)$
Multiply.	$-(8 \cdot 2)$
Multiply.	-16

Note:
Exercise:

Problem: Simplify:

ⓐ $(-3)^4$

ⓑ -3^4

Solution:

ⓐ 81
ⓑ −81

Note:
Exercise:

Problem: Simplify:

ⓐ $(-7)^2$
ⓑ -7^2

Solution:

ⓐ 49
ⓑ −49

Example:
Exercise:

Problem: Simplify: $12 - 3(9 - 12)$.

Solution:
Solution

According to the order of operations, we simplify inside parentheses first. Then we will multiply and finally we will subtract.

	$12 - 3(9 - 12)$
Subtract the parentheses first.	$12 - 3(-3)$
Multiply.	$12 - (-9)$
Subtract.	21

Note:
Exercise:

Problem: Simplify:

$17 - 4(8 - 11)$

Solution:

29

Note:
Exercise:

Problem: Simplify:

$16 - 6(7 - 13)$

Solution:

52

Example:
Exercise:

Problem: Simplify: $8(-9) \div (-2)^3$.

Solution:
Solution

We simplify the exponent first, then multiply and divide.

	$8(-9) \div (-2)^3$
Simplify the exponent.	$8(-9) \div (-8)$
Multiply.	$-72 \div (-8)$
Divide.	9

Note:
Exercise:

Problem: Simplify:

$12(-9) \div (-3)^3$

Solution:

4

Note:
Exercise:

Problem: Simplify:

$18(-4) \div (-2)^3$

Solution:

9

Example:
Exercise:

Problem: Simplify: $-30 \div 2 + (-3)(-7)$.

Solution:
Solution

First we will multiply and divide from left to right. Then we will add.

	$-30 \div 2 + (-3)(-7)$
Divide.	$-15 + (-3)(-7)$
Multiply.	$-15 + 21$
Add.	6

Note:
Exercise:

Problem: Simplify:

$-27 \div 3 + (-5)(-6)$

Solution:

21

Note:
Exercise:

Problem: Simplify:

$-32 \div 4 + (-2)(-7)$

Solution:

6

Evaluate Variable Expressions with Integers

Now we can evaluate expressions that include multiplication and division with integers. Remember that to evaluate an expression, substitute the numbers in place of the variables, and then simplify.

Example:
Exercise:

Problem: Evaluate $2x^2 - 3x + 8$ when $x = -4$.

Solution:
Solution

	$2x^2 - 3x + 8$
Substitute -4 for x.	$2(-4)^2 - 3(-4) + 8$
Simplify exponents.	$2(16) - 3(-4) + 8$
Multiply.	$32 - (-12) + 8$
Subtract.	$44 + 8$
Add.	52

Keep in mind that when we substitute -4 for x, we use parentheses to show the multiplication. Without parentheses, it would look like $2 \cdot -4^2 - 3 \cdot -4 + 8$.

Note:
Exercise:

Problem: Evaluate:

$3x^2 - 2x + 6$ when $x = -3$

Solution:

39

Note:
Exercise:

Problem: Evaluate:

$4x^2 - x - 5$ when $x = -2$

Solution:

13

Example:
Exercise:

Problem: Evaluate $3x + 4y - 6$ when $x = -1$ and $y = 2$.

Solution:
Solution

	$3x + 4y - 6$

Substitute $x = -1$ and $y = 2$.	$3(-1) + 4(2) - 6$
Multiply.	$-3 + 8 - 6$
Simplify.	-1

Note:
Exercise:

Problem: Evaluate:

$7x + 6y - 12$ when $x = -2$ and $y = 3$

Solution:

−8

Note:
Exercise:

Problem: Evaluate:

$8x - 6y + 13$ when $x = -3$ and $y = -5$

Solution:

19

Translate Word Phrases to Algebraic Expressions

Once again, all our prior work translating words to algebra transfers to phrases that include both multiplying and dividing integers. Remember that the key word for multiplication is *product* and for division is *quotient*.

Example:
Exercise:

Problem:

Translate to an algebraic expression and simplify if possible: the product of -2 and 14.

Solution:
Solution

The word *product* tells us to multiply.

	the product of -2 and 14
Translate.	$(-2)(14)$
Simplify.	-28

Note:
Exercise:

Problem: Translate to an algebraic expression and simplify if possible:

the product of -5 and 12

Solution:

−5 (12) = −60

Note:
Exercise:

Problem: Translate to an algebraic expression and simplify if possible:

the product of 8 and -13

Solution:

8 (−13) = −104

Example:
Exercise:

Problem:

Translate to an algebraic expression and simplify if possible: the quotient of -56 and -7.

Solution:
Solution

The word *quotient* tells us to divide.

	the quotient of −56 and −7

Translate.	$-56 \div (-7)$
Simplify.	8

Note:
Exercise:

Problem: Translate to an algebraic expression and simplify if possible:

the quotient of -63 and -9

Solution:

−63 ÷ −9 = 7

Note:
Exercise:

Problem: Translate to an algebraic expression and simplify if possible:

the quotient of -72 and -9

Solution:

−72 ÷ −9 = 8

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Multiplying Integers Using Color Counters
- Multiplying Integers Using Color Counters With Neutral Pairs

- Multiplying Integers Basics
- Dividing Integers Basics
- Ex. Dividing Integers
- Multiplying and Dividing Signed Numbers

Key Concepts

- **Multiplication of Signed Numbers**
    - To determine the sign of the product of two signed numbers:

Same Signs	Product
Two positives Two negatives	Positive Positive

Different Signs	Product
Positive • negative Negative • positive	Negative Negative

- **Division of Signed Numbers**
    - To determine the sign of the quotient of two signed numbers:

Same Signs	Quotient
Two positives Two negatives	Positive Positive

Different Signs	Quotient
Positive • negative Negative • Positive	Negative Negative

- **Multiplication by** -1
    - Multiplying a number by -1 gives its opposite: $-1a = -a$
- **Division by** -1
    - Dividing a number by -1 gives its opposite: $a \div (-1) = -a$

Practice Makes Perfect

Multiply Integers

In the following exercises, multiply each pair of integers.
Exercise:

Problem: $-4 \cdot 8$

Solution:

−32

Exercise:

Problem: $-3 \cdot 9$

Exercise:

Problem: $-5(7)$

Solution:

-35

Exercise:

Problem: $-8(6)$

Exercise:

Problem: $-18(-2)$

Solution:

36

Exercise:

Problem: $-10(-6)$

Exercise:

Problem: $9(-7)$

Solution:

-63

Exercise:

Problem: $13(-5)$

Exercise:

Problem: $-1 \cdot 6$

Solution:

-6

Exercise:

Problem: $-1 \cdot 3$

Exercise:

Problem: $-1(-14)$

Solution:

14

Exercise:

Problem: $-1(-19)$

Divide Integers

In the following exercises, divide.

Exercise:

Problem: $-24 \div 6$

Solution:

-4

Exercise:

Problem: $-28 \div 7$

Exercise:

Problem: $56 \div (-7)$

Solution:

-8

Exercise:

Problem: $35 \div (-7)$

Exercise:

Problem: $-52 \div (-4)$

Solution:

13

Exercise:

Problem: $-84 \div (-6)$

Exercise:

Problem: $-180 \div 15$

Solution:

-12

Exercise:

Problem: $-192 \div 12$

Exercise:

Problem: $49 \div (-1)$

Solution:

-49

Exercise:

Problem: $62 \div (-1)$

Simplify Expressions with Integers

In the following exercises, simplify each expression.

Exercise:

Problem: $5(-6) + 7(-2) - 3$

Solution:

-47

Exercise:

Problem: $8(-4) + 5(-4) - 6$

Exercise:

Problem: $-8(-2) - 3(-9)$

Solution:

43

Exercise:

Problem: $-7(-4) - 5(-3)$

Exercise:

Problem: $(-5)^3$

Solution:

-125

Exercise:

Problem: $(-4)^3$

Exercise:

Problem: $(-2)^6$

Solution:

64

Exercise:

Problem: $(-3)^5$

Exercise:

Problem: -4^2

Solution:

-16

Exercise:

Problem: -6^2

Exercise:

Problem: $-3(-5)(6)$

Solution:

90

Exercise:

Problem: $-4(-6)(3)$

Exercise:

Problem: $-4 \cdot 2 \cdot 11$

Solution:

-88

Exercise:

Problem: $-5 \cdot 3 \cdot 10$

Exercise:

Problem: $(8-11)(9-12)$

Solution:

9

Exercise:

Problem: $(6-11)(8-13)$

Exercise:

Problem: $26-3(2-7)$

Solution:

41

Exercise:

Problem: $23-2(4-6)$

Exercise:

Problem: $-10(-4) \div (-8)$

Solution:

-5

Exercise:

Problem: $-8(-6) \div (-4)$

Exercise:

Problem: $65 \div (-5) + (-28) \div (-7)$

Solution:

-9

Exercise:

Problem: $52 \div (-4) + (-32) \div (-8)$

Exercise:

Problem: $9 - 2[3 - 8(-2)]$

Solution:

-29

Exercise:

Problem: $11 - 3[7 - 4(-2)]$

Exercise:

Problem: $(-3)^2 - 24 \div (8 - 2)$

Solution:

5

Exercise:

Problem: $(-4)^2 - 32 \div (12 - 4)$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.
Exercise:

Problem: $-2x + 17$ when

ⓐ $x = 8$
ⓑ $x = -8$

Solution:

ⓐ 1
ⓑ 33

Exercise:

Problem: $-5y + 14$ when

ⓐ $y = 9$
ⓑ $y = -9$

Exercise:

Problem: $10 - 3m$ when

ⓐ $m = 5$
ⓑ $m = -5$

Solution:

ⓐ −5
ⓑ 25

Exercise:

Problem: $18 - 4n$ when

ⓐ $n = 3$
ⓑ $n = -3$

Exercise:

Problem: $p^2 - 5p + 5$ when $p = -1$

Solution:

8

Exercise:

Problem: $q^2 - 2q + 9$ when $q = -2$

Exercise:

Problem: $2w^2 - 3w + 7$ when $w = -2$

Solution:

21

Exercise:

Problem: $3u^2 - 4u + 5$ when $u = -3$

Exercise:

Problem: $6x - 5y + 15$ when $x = 3$ and $y = -1$

Solution:

38

Exercise:

Problem: $3p - 2q + 9$ when $p = 8$ and $q = -2$

Exercise:

Problem: $9a - 2b - 8$ when $a = -6$ and $b = -3$

Solution:

−56

Exercise:

Problem: $7m - 4n - 2$ when $m = -4$ and $n = -9$

Translate Word Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

Exercise:

Problem: The product of -3 and 15

Solution:

$-3 \cdot 15 = -45$

Exercise:

Problem: The product of -4 and 16

Exercise:

Problem: The quotient of -60 and -20

Solution:

$-60 \div (-20) = 3$

Exercise:

Problem: The quotient of -40 and -20

Exercise:

Problem: The quotient of -6 and the sum of a and b

Solution:

$\frac{-6}{a+b}$

Exercise:

Problem: The quotient of -7 and the sum of m and n

Exercise:

Problem: The product of -10 and the difference of p and q

Solution:

$-10\,(p - q)$

Exercise:

Problem: The product of -13 and the difference of c and d

Everyday Math

Exercise:

Problem:

Stock market Javier owns 300 shares of stock in one company. On Tuesday, the stock price dropped \$12 per share. What was the total effect on Javier's portfolio?

Solution:

−$3,600

Exercise:

Problem:

Weight loss In the first week of a diet program, eight women lost an average of 3 pounds each. What was the total weight change for the eight women?

Writing Exercises

Exercise:

Problem: In your own words, state the rules for multiplying two integers.

Solution:

Sample answer: Multiplying two integers with the same sign results in a positive product. Multiplying two integers with different signs results in a negative product.

Exercise:

Problem: In your own words, state the rules for dividing two integers.

Exercise:

Problem: Why is $-2^4 \neq (-2)^4$?

Solution:

Sample answer: In one expression the base is positive and then we take the opposite, but in the other the base is negative.

Exercise:

Problem: Why is $-4^2 \neq (-4)^2$?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
multiply integers.			
divide integers.			
simplify expressions with integers.			
evaluate variable expressions with integers.			
translate word phrases to algebraic expressions.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Solve Equations Using Integers; The Division Property of Equality
By the end of this section, you will be able to:

- Determine whether an integer is a solution of an equation
- Solve equations with integers using the Addition and Subtraction Properties of Equality
- Model the Division Property of Equality
- Solve equations using the Division Property of Equality
- Translate to an equation and solve

Note:
Before you get started, take this readiness quiz.

1. Evaluate $x+4$ when $x=-4$.
   If you missed this problem, review [link].
2. Solve: $y-6=10$.
   If you missed this problem, review [link].
3. Translate into an algebraic expression *5 less than* x.
   If you missed this problem, review [link].

Determine Whether a Number is a Solution of an Equation

In Solve Equations Using the Subtraction and Addition Properties of Equality, we saw that a solution of an equation is a value of a variable that makes a true statement when substituted into that equation. In that section, we found solutions that were whole numbers. Now that we've worked with integers, we'll find integer solutions to equations.

The steps we take to determine whether a number is a solution to an equation are the same whether the solution is a whole number or an integer.

Note:
How to determine whether a number is a solution to an equation.

Substitute the number for the variable in the equation.
Simplify the expressions on both sides of the equation.
Determine whether the resulting equation is true.

- If it is true, the number is a solution.
- If it is not true, the number is not a solution.

Example:
Exercise:

Problem:

Determine whether each of the following is a solution of $2x - 5 = -13$:

ⓐ $x = 4$
ⓑ $x = -4$
ⓒ $x = -9$.

Solution:
Solution

ⓐ Substitute 4 for x in the equation to determine if it is true.	

	$2x - 5 = -13$
Substitute 4 for x.	$2(4) - 5 \stackrel{?}{=} -13$
Multiply.	$8 - 5 \stackrel{?}{=} -13$
Subtract.	$3 \neq -13$

Since $x = 4$ does not result in a true equation, 4 is not a solution to the equation.

ⓑ Substitute −4 for x in the equation to determine if it is true.	$2x - 5 = -13$
Substitute −4 for x.	$2(-4) - 5 \stackrel{?}{=} -13$
Multiply.	$-8 - 5 \stackrel{?}{=} -13$
Subtract.	$-13 = -13$ ✓

Since $x = -4$ results in a true equation, -4 is a solution to the equation.

ⓒ Substitute −9 for x in the equation to determine if it is true.	
	$2x - 5 = -13$
Substitute −9 for x.	$2(-9) - 5 \stackrel{?}{=} -13$
Multiply.	$-18 - 5 \stackrel{?}{=} -13$
Subtract.	$-23 \neq -13$

Since $x = -9$ does not result in a true equation, -9 is not a solution to the equation.

Note:
Exercise:

Problem:

Determine whether each of the following is a solution of $2x - 8 = -14$:

ⓐ $x = -11$
ⓑ $x = 11$
ⓒ $x = -3$

Solution:

ⓐ no
ⓑ no
ⓒ yes

Note:
Exercise:

Problem:

Determine whether each of the following is a solution of $2y + 3 = -11$:

ⓐ $y = 4$
ⓑ $y = -4$
ⓒ $y = -7$

Solution:

ⓐ no
ⓑ no
ⓒ yes

Solve Equations with Integers Using the Subtraction and Addition Properties of Equality

In Solve Equations with the Subtraction and Addition Properties of Equality, we solved equations similar to the two shown here using the Subtraction and Addition Properties of Equality. Now we can use them again with integers.

$$x + 4 = 12 \qquad\qquad y - 5 = 9$$

$$x + 4 - 4 = 12 - 4 \qquad\qquad y - 5 + 5 = 9 + 5$$

$$x = 8 \qquad\qquad y = 14$$

When you add or subtract the same quantity from both sides of an equation, you still have equality.

Note:
Properties of Equalities

Subtraction Property of Equality	**Addition Property of Equality**
For any numbers a, b, c, if $a = b$ then $a - c = b - c$.	For any numbers a, b, c, if $a = b$ then $a + c = b + c$.

Note that we actually don't need the Subtraction Property of Equality since any subtraction can be accomplished by adding the opposite value. For any real numbers a and b, a - b = a + (-b).

Example:
Exercise:

Problem: Solve: $y + 9 = 5$.

Solution:
Solution

	$y+9=5$
Subtract 9 from each side to undo the addition.	$y+9-9=5-9$
Simplify.	$y=-4$

Check the result by substituting -4 into the original equation.

	$y+9=5$
Substitute −4 for y	$-4+9\overset{?}{=}5$
	$5=5\checkmark$

Since $y = -4$ makes $y + 9 = 5$ a true statement, we found the solution to this equation.

Note:
Exercise:

Problem: Solve:

$y + 11 = 7$

Solution:

−4

Note:
Exercise:

Problem: Solve:

$y + 15 = -4$

Solution:

−19

Example:
Exercise:

Problem: Solve: $a - 6 = -8$

Solution:
Solution

	$a-6=-8$
Add 6 to each side to undo the subtraction.	$a-6+6=-8+6$
Simplify.	$a=-2$
Check the result by substituting -2 into the original equation:	$a-6=-8$
Substitute -2 for a	$-2-6\stackrel{?}{=}-8$
	$-8=-8\checkmark$

The solution to $a-6=-8$ is -2.

Since $a=-2$ makes $a-6=-8$ a true statement, we found the solution to this equation.

Note:

Exercise:

Problem: Solve:

$a - 2 = -8$

Solution:

−6

Note:
Exercise:

Problem: Solve:

$n - 4 = -8$

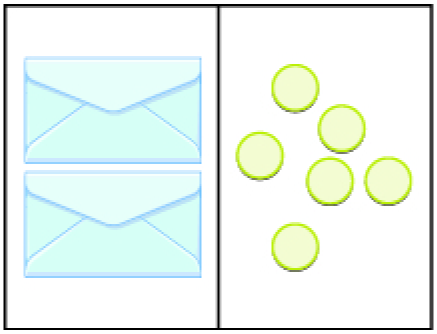
Solution:

−4

Model the Division Property of Equality

All of the equations we have solved so far have been of the form $x + a = b$ or $x - a = b$. We were able to isolate the variable by adding or subtracting the constant term. Now we'll see how to solve equations that involve division.

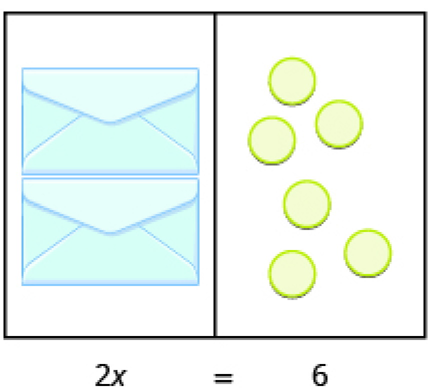
We will model an equation with envelopes and counters in [link].



Here, there are two identical envelopes that contain the same number of counters. Remember, the left side of the workspace must equal the right side, but the counters on the left side are “hidden” in the envelopes. So how many counters are in each envelope?

To determine the number, separate the counters on the right side into 2 groups of the same size. So 6 counters divided into 2 groups means there must be 3 counters in each group (since $6 \div 2 = 3$).

What equation models the situation shown in [link]? There are two envelopes, and each contains x counters. Together, the two envelopes must contain a total of 6 counters. So the equation that models the situation is $2x = 6$.

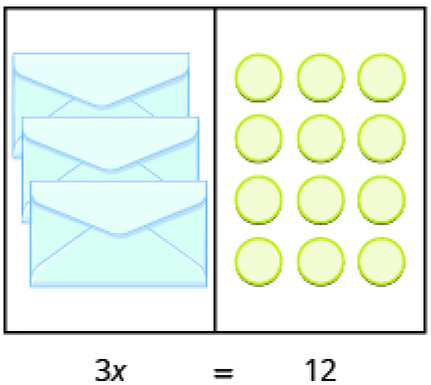


We can divide both sides of the equation by 2 as we did with the envelopes and counters.

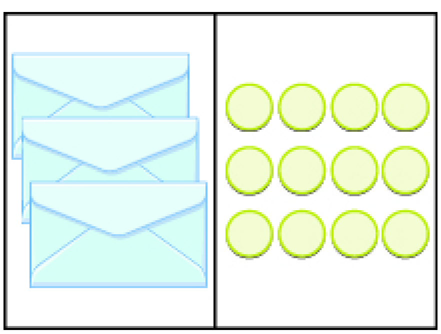
$$\frac{2x}{2} = \frac{6}{2}$$
$$x = 3$$

We found that each envelope contains 3 counters. Does this check? We know $2 \cdot 3 = 6$, so it works. Three counters in each of two envelopes does equal six.

[link] shows another example, this time where the counters are negative.



Now we have 3 identical envelopes and 12 negative counters. How many negative counters are in each envelope? We have to separate the 12 negative counters into 3 groups. Since $-12 \div 3 = -4$, there must be 4 negative counters in each envelope. See [link].



The equation that models the situation is $3x = -12$. We can divide both sides of the equation by 3.

$$\frac{3x}{3} = \frac{12}{3}$$

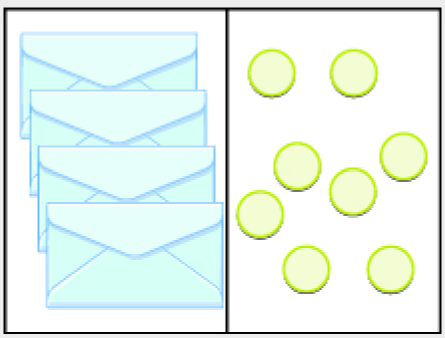
$$x = 4$$

Does this check? It does because $3 \cdot (\text{-}4) = -12$.

Example:
Exercise:

Problem:

Write an equation modeled by the envelopes and counters, and then solve it.



Solution:
Solution

There are 4 envelopes, or 4 unknown values, on the left that match the 8 counters on the right. Let's call the unknown quantity in the envelopes x.

Write the equation.	$4x = 8$

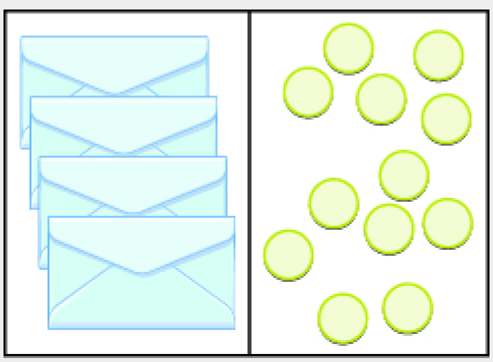
Divide both sides by 4.	$\frac{4x}{4} = \frac{8}{4}$
Simplify.	$x = 2$

There are 2 counters in each envelope.

Note:
Exercise:

Problem:

Write the equation modeled by the envelopes and counters. Then solve it.



Solution:

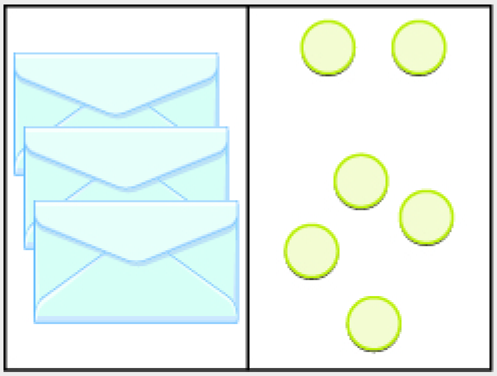
$4x = 12$; $x = 3$

Note:

Exercise:

Problem:

Write the equation modeled by the envelopes and counters. Then solve it.



Solution:

$3x = 6$; $x = 2$

Solve Equations Using the Division Property of Equality

The previous examples lead to the Division Property of Equality. When you divide both sides of an equation by any nonzero number, you still have equality.

Note:
Division Property of Equality
Equation:

$$\begin{array}{rl} \text{For any numbers} \quad a, b, c, \text{and} & c \neq 0, \\ \text{If} \quad a = b \text{ then} & \frac{a}{c} = \frac{b}{c}. \end{array}$$

Example:
Exercise:

Problem: Solve: $7x = -49$.

Solution:
Solution

To isolate x, we need to undo multiplication.

	$7x = -49$
Divide each side by 7.	$\frac{7x}{7} = \frac{-49}{7}$
Simplify.	$x = -7$

Check the solution.

	$7x = -49$
Substitute −7 for x.	$7(-7) \stackrel{?}{=} -49$

	$-49 = -49\checkmark$

Therefore, -7 is the solution to the equation.

Note:
Exercise:

Problem: Solve:

$8a = 56$

Solution:

7

Note:
Exercise:

Problem: Solve:

$11n = 121$

Solution:

11

Example:
Exercise:

Problem: Solve: $-3y = 63$.

Solution:
Solution

To isolate y, we need to undo the multiplication.

	$-3y = 63$
Divide each side by −3.	$\frac{-3y}{-3} = \frac{63}{-3}$
Simplify	$y = -21$

Check the solution.

	$-3y = 63$
Substitute −21 for y.	$-3(-21) \stackrel{?}{=} 63$

	$63 = 63\checkmark$

Since this is a true statement, $y = -21$ is the solution to the equation. This was our first example with a negative coefficient, -3. It is hard to think of this as putting 63 positive counters in negative 3 envelopes. After all, what is a negative envelope? While dividing both sides of the equation by -3 quickly led to the solution, why does that work? One way to justify the solution is to notice that if a equals b then the opposite of a equals the opposite of b: if a = b then -a = -b. In this example, -3y = 63 then -(-3y)= -63. The opposite of -3y is 3y, so the equation simplifies to 3y = -63 and this equation has a positive coefficient and can be solved just as before. Thankfully, we don't have to go through all of these steps each time when we have a negative coefficient, we just divide by it. But now we have a better feeling for why this works.

Note:
Exercise:

Problem: Solve:

$-8p = 96$

Solution:

−12

Note:
Exercise:

Problem: Solve:

$-12m = 108$

Solution:

-9

Translate to an Equation and Solve

In the past several examples, we were given an equation containing a variable. In the next few examples, we'll have to first translate word sentences into equations with variables and then we will solve the equations.

Example:
Exercise:

Problem: Translate and solve: five more than x is equal to -3.

Solution:
Solution

	five more than x is equal to -3
Translate	$x + 5 = -3$
Subtract 5 from both sides.	$x + 5 - 5 = -3 - 5$
Simplify.	$x = -8$

Check the answer by substituting it into the original equation.

$$\begin{aligned} x + 5 &= -3 \\ -8 + 5 &\stackrel{?}{=} -3 \\ -3 &= -3 \checkmark \end{aligned}$$

Note:
Exercise:

Problem: Translate and solve:

Seven more than x is equal to -2.

Solution:

$x + 7 = -2$; $x = -9$

Note:
Exercise:

Problem: Translate and solve:

Eleven more than y is equal to 2.

Solution:

$y + 11 = 2$; $y = -9$

Example:
Exercise:

Problem: Translate and solve: the difference of n and 6 is -10.

Solution:
Solution

	the difference of n and 6 is -10
Translate.	$n - 6 = -10$
Add 6 to each side.	$n - 6 + 6 = -10 + 6$
Simplify.	$n = -4$

Check the answer by substituting it into the original equation.

$$\begin{aligned} n - 6 &= -10 \\ -4 - 6 &\stackrel{?}{=} -10 \\ -10 &= -10 \checkmark \end{aligned}$$

Note:
Exercise:

Problem: Translate and solve:

The difference of p and 2 is -4.

Solution:

$p - 2 = -4$; $p = -2$

Note:
Exercise:

Problem: Translate and solve:

The difference of q and 7 is -3.

Solution:

$q - 7 = -3$; $q = 4$

Example:
Exercise:

Problem:

Translate and solve: the number 108 is the product of -9 and y.

Solution:
Solution

	the number of 108 is the product of -9 and y

Translate.	$108 = -9y$
Divide by -9 .	$\frac{108}{-9} = \frac{-9y}{-9}$
Simplify.	$-12 = y$

Check the answer by substituting it into the original equation.

$108 = -9y$

$108 \stackrel{?}{=} -9(-12)$

$108 = 108\checkmark$

Note:
Exercise:

Problem: Translate and solve:

The number 132 is the product of -12 and y.

Solution:

$132 = -12y$; $y = -11$

Note:
Exercise:

Problem: Translate and solve:

The number 117 is the product of -13 and z.

Solution:

117 = −13z; z = −9

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- One-Step Equations With Adding Or Subtracting
- One-Step Equations With Multiplying Or Dividing

Key Concepts

- **How to determine whether a number is a solution to an equation.**
    - Step 1. Substitute the number for the variable in the equation.
    - Step 2. Simplify the expressions on both sides of the equation.
    - Step 3. Determine whether the resulting equation is true.

      If it is true, the number is a solution.
      If it is not true, the number is not a solution.
- **Properties of Equalities**

Subtraction Property of Equality	Addition Property of Equality

Subtraction Property of Equality	Addition Property of Equality
For any numbers a, b, c, if $a = b$ then $a - c = b - c$.	For any numbers a, b, c, if $a = b$ then $a + c = b + c$.

- **Division Property of Equality**
  - For any numbers a, b, c, and $c \neq 0$
    If $a = b$, then $\frac{a}{c} = \frac{b}{c}$.

Section Exercises

Practice Makes Perfect

Determine Whether a Number is a Solution of an Equation

In the following exercises, determine whether each number is a solution of the given equation.

Exercise:

Problem: $4x - 2 = 6$

ⓐ $x = -2$
ⓑ $x = -1$
ⓒ $x = 2$

Solution:

ⓐ no
ⓑ no
ⓒ yes

Exercise:

Problem: $4y - 10 = -14$

ⓐ $y = -6$
ⓑ $y = -1$
ⓒ $y = 1$

Exercise:

Problem: $9a + 27 = -63$

ⓐ $a = 6$
ⓑ $a = -6$
ⓒ $a = -10$

Solution:

ⓐ no
ⓑ no
ⓒ yes

Exercise:

Problem: $7c + 42 = -56$

ⓐ $c = 2$
ⓑ $c = -2$
ⓒ $c = -14$

Solve Equations Using the Addition and Subtraction Properties of Equality

In the following exercises, solve for the unknown.
Exercise:

Problem: $n + 12 = 5$

Solution:

-7

Exercise:

Problem: $m + 16 = 2$

Exercise:

Problem: $p + 9 = -8$

Solution:

-17

Exercise:

Problem: $q + 5 = -6$

Exercise:

Problem: $u - 3 = -7$

Solution:

-4

Exercise:

Problem: $v - 7 = -8$

Exercise:

Problem: $h - 10 = -4$

Solution:

6

Exercise:

Problem: $k - 9 = -5$

Exercise:

Problem: $x + (-2) = -18$

Solution:

−16

Exercise:

Problem: $y + (-3) = -10$

Exercise:

Problem: $r - (-5) = -9$

Solution:

−14

Exercise:

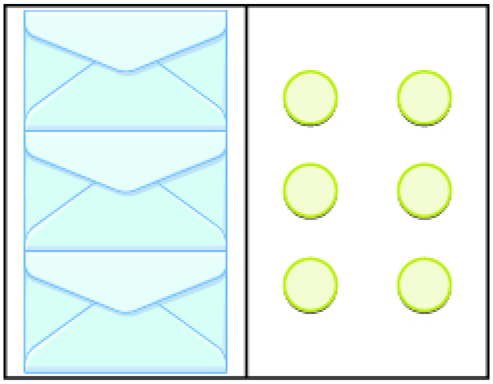
Problem: $s - (-2) = -11$

Model the Division Property of Equality

In the following exercises, write the equation modeled by the envelopes and counters and then solve it.

Exercise:

Problem:

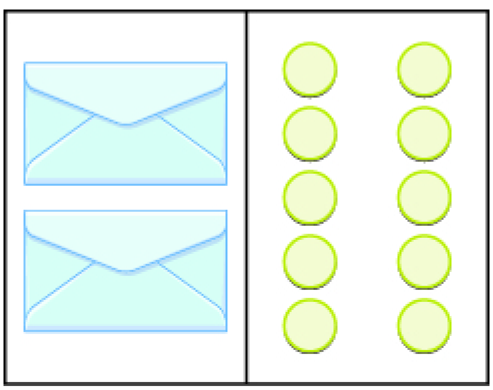


Solution:

$3x = 6$; $x = 2$

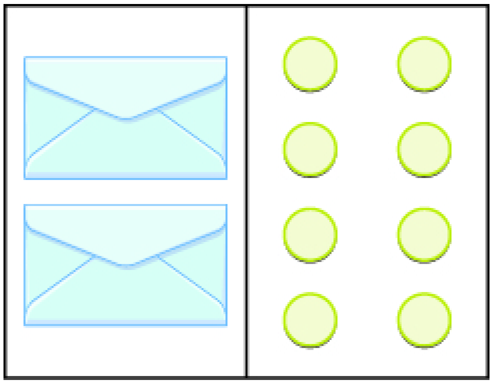
Exercise:

Problem:



Exercise:

Problem:

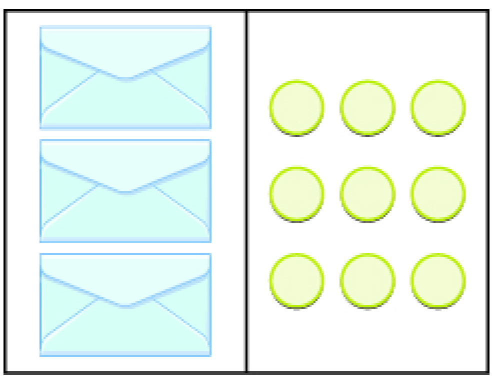


Solution:

$2x = 8$; $x = 4$

Exercise:

Problem:



Solve Equations Using the Division Property of Equality

In the following exercises, solve each equation using the division property of equality and check the solution.

Exercise:

Problem: $5x = 45$

Solution:

9

Exercise:

Problem: $4p = 64$

Exercise:

Problem: $-7c = 56$

Solution:

−8

Exercise:

Problem: $-9x = 54$

Exercise:

Problem: $-14p = -42$

Solution:

3

Exercise:

Problem: $-8m = -40$

Exercise:

Problem: $-120 = 10q$

Solution:

−12

Exercise:

Problem: $-75 = 15y$

Exercise:

Problem: $24x = 480$

Solution:

20

Exercise:

Problem: $18n = 540$

Exercise:

Problem: $-3z = 0$

Solution:

0

Exercise:

Problem: $4u = 0$

Translate to an Equation and Solve

In the following exercises, translate and solve.

Exercise:

Problem: Four more than n is equal to 1.

Solution:

$n + 4 = 1$; $n = -3$

Exercise:

Problem: Nine more than m is equal to 5.

Exercise:

Problem: The sum of eight and p is -3.

Solution:

$8 + p = -3$; $p = -11$

Exercise:

Problem: The sum of two and q is -7.

Exercise:

Problem: The difference of a and three is -14.

Solution:

$a - 3 = -14$; $a = -11$

Exercise:

Problem: The difference of b and 5 is -2.

Exercise:

Problem: The number -42 is the product of -7 and x.

Solution:

$-42 = -7x$; $x = 6$

Exercise:

Problem: The number -54 is the product of -9 and y.

Exercise:

Problem: The product of f and -15 is 75.

Solution:

$f(-15) = 75$; $f = 5$

Exercise:

Problem: The product of g and -18 is 36.

Exercise:

Problem: -6 plus c is equal to 4.

Solution:

$-6 + c = 4$; $c = 10$

Exercise:

Problem: -2 plus d is equal to 1.

Exercise:

Problem: Nine less than n is -4.

Solution:

$m - 9 = -4$; $m = 5$

Exercise:

Problem: Thirteen less than n is -10.

Mixed Practice

In the following exercises, solve.

Exercise:

Problem:

ⓐ $x+2=10$
ⓑ $2x=10$

Solution:

ⓐ 8
ⓑ 5

Exercise:

Problem:

ⓐ $y+6=12$
ⓑ $6y=12$

Exercise:

Problem:

ⓐ $-3p=27$
ⓑ $p-3=27$

Solution:

ⓐ −9
ⓑ 30

Exercise:

Problem:

ⓐ $-2q=34$

ⓑ $q - 2 = 34$

Exercise:

Problem: $a - 4 = 16$

Solution:

20

Exercise:

Problem: $b - 1 = 11$

Exercise:

Problem: $-8m = -56$

Solution:

7

Exercise:

Problem: $-6n = -48$

Exercise:

Problem: $-39 = u + 13$

Solution:

−52

Exercise:

Problem: $-100 = v + 25$

Exercise:

Problem: $11r = -99$

Solution:

-9

Exercise:

Problem: $15s = -300$

Exercise:

Problem: $100 = 20d$

Solution:

5

Exercise:

Problem: $250 = 25n$

Exercise:

Problem: $-49 = x - 7$

Solution:

-42

Exercise:

Problem: $64 = y - 4$

Everyday Math

Exercise:

Problem:

Cookie packaging A package of 51 cookies has 3 equal rows of cookies. Find the number of cookies in each row, c, by solving the equation $3c = 51$.

Solution:

17 cookies

Exercise:

Problem:

Kindergarten class Connie's kindergarten class has 24 children. She wants them to get into 4 equal groups. Find the number of children in each group, g, by solving the equation $4g = 24$.

Writing Exercises

Exercise:

Problem:

Is modeling the Division Property of Equality with envelopes and counters helpful to understanding how to solve the equation $3x = 15$? Explain why or why not.

Solution:

Sample answer: It is helpful because it shows how the counters can be divided among the envelopes.

Exercise:

Problem:

Suppose you are using envelopes and counters to model solving the equations $x + 4 = 12$ and $4x = 12$. Explain how you would solve each equation.

Exercise:

Problem:

Frida started to solve the equation $-3x = 36$ by adding 3 to both sides. Explain why Frida's method will not solve the equation.

Solution:

Sample answer: The operation used in the equation is multiplication. The inverse of multiplication is division, not addition.

Exercise:

Problem:

Raoul started to solve the equation $4y = 40$ by subtracting 4 from both sides. Explain why Raoul's method will not solve the equation.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
determine whether an integer is a solution of an equation.			
solve equations with integers using the addition and subtraction properties of equality.			
model division property of equality.			
solve equations using the division property of equality.			
translate to an equation and solve.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next Chapter? Why or why not?

Chapter Review Exercises

Introduction to Integers

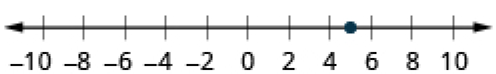
Locate Positive and Negative Numbers on the Number Line

In the following exercises, locate and label the integer on the number line.

Exercise:

Problem: 5

Solution:



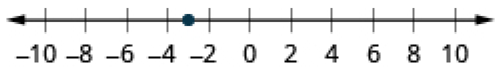
Exercise:

Problem: -5

Exercise:

Problem: -3

Solution:



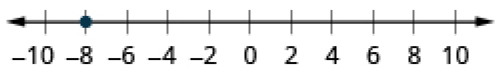
Exercise:

Problem: 3

Exercise:

Problem: -8

Solution:



Exercise:

Problem: -7

Order Positive and Negative Numbers

In the following exercises, order each of the following pairs of numbers, using $<$ or $>$.

Exercise:

Problem: $4\_\_8$

Solution:

$<$

Exercise:

Problem: $-6\_\_3$

Exercise:

Problem: $-5\_\_-10$

Solution:

$>$

Exercise:

Problem: $-9\_\_-4$

Exercise:

Problem: $2\_\_-7$

Solution:

$>$

Exercise:

Problem: $-3\_\_1$

Find Opposites

In the following exercises, find the opposite of each number.

Exercise:

Problem: 6

Solution:

-6

Exercise:

Problem: -2

Exercise:

Problem: -4

Solution:

4

Exercise:

Problem: 3

In the following exercises, simplify.

Exercise:

Problem:

ⓐ $-(8)$
ⓑ $-(-8)$

Solution:

ⓐ -8
ⓑ 8

Exercise:

Problem:

ⓐ $-(9)$
ⓑ $-(-9)$

In the following exercises, evaluate.
Exercise:

Problem: $-x,$ when

ⓐ $x=32$
ⓑ $x=-32$

Solution:

ⓐ −32
ⓑ 32

Exercise:

Problem: $-n,$ when

ⓐ $n=20$
ⓐ $n=-20$

Simplify Absolute Values

In the following exercises, simplify.
Exercise:

Problem: $|-21|$

Solution:

21

Exercise:

Problem: $|-42|$

Exercise:

Problem: $|36|$

Solution:

36

Exercise:

Problem: $-|15|$

Exercise:

Problem: $|0|$

Solution:

0

Exercise:

Problem: $-|-75|$

In the following exercises, evaluate.

Exercise:

Problem: $|x|$ when $x=-14$

Solution:

14

Exercise:

Problem: $-|r|$ when $r = 27$

Exercise:

Problem: $-|-y|$ when $y = 33$

Solution:

−33

Exercise:

Problem: $|-n|$ when $n = -4$

In the following exercises, fill in $<$, $>$, or $=$ for each of the following pairs of numbers.

Exercise:

Problem: $-|-4|\_\_4$

Solution:

$<$

Exercise:

Problem: $-2\_\_|-2|$

Exercise:

Problem: $-|-6|\_\_-6$

Solution:

$=$

Exercise:

Problem: $-|-9|\_\_|-9|$

In the following exercises, simplify.

Exercise:

Problem: $-(-55)$ and $-|-55|$

Solution:

-55; -55

Exercise:

Problem: $-(-48)$ and $-|-48|$

Exercise:

Problem: $|12-5|$

Solution:

7

Exercise:

Problem: $|9+7|$

Exercise:

Problem: $6|-9|$

Solution:

54

Exercise:

Problem: $|14-8| - |-2|$

Exercise:

Problem: $|9-3| - |5-12|$

Solution:

-1

Exercise:

Problem: $5 + 4\,|15-3|$

Translate Phrases to Expressions with Integers

In the following exercises, translate each of the following phrases into expressions with positive or negative numbers.

Exercise:

Problem: the opposite of 16

Solution:

-16

Exercise:

Problem: the opposite of -8

Exercise:

Problem: negative 3

Solution:

-3

Exercise:

Problem: 19 minus negative 12

Exercise:

Problem: a temperature of 10 below zero

Solution:

$-10°$

Exercise:

Problem: an elevation of 85 feet below sea level

Add Integers

Model Addition of Integers

In the following exercises, model the following to find the sum.

Exercise:

Problem: $3 + 7$

Solution:

10

Exercise:

Problem: $-2 + 6$

Exercise:

Problem: $5 + (-4)$

Solution:

1

Exercise:

Problem: $-3 + (-6)$

Simplify Expressions with Integers

In the following exercises, simplify each expression.

Exercise:

Problem: $14 + 82$

Solution:

96

Exercise:

Problem: $-33 + (-67)$

Exercise:

Problem: $-75 + 25$

Solution:

−50

Exercise:

Problem: $54 + (-28)$

Exercise:

Problem: $11 + (-15) + 3$

Solution:

-1

Exercise:

Problem: $-19 + (-42) + 12$

Exercise:

Problem: $-3 + 6(-1 + 5)$

Solution:

21

Exercise:

Problem: $10 + 4(-3 + 7)$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

Exercise:

Problem: $n + 4$ when

ⓐ $n = -1$
ⓑ $n = -20$

Solution:

ⓐ 3
ⓑ -16

Exercise:

Problem: $x + (-9)$ when

ⓐ $x = 3$
ⓑ $x = -3$

Exercise:

Problem: $(x + y)^3$ when $x = -4, y = 1$

Solution:

−27

Exercise:

Problem: $(u + v)^2$ when $u = -4, v = 11$

Translate Word Phrases to Algebraic Expressions

In the following exercises, translate each phrase into an algebraic expression and then simplify.

Exercise:

Problem: the sum of -8 and 2

Solution:

−8 + 2 = −6

Exercise:

Problem: 4 more than -12

Exercise:

Problem: 10 more than the sum of -5 and -6

Solution:

$10 + [-5 + (-6)] = -1$

Exercise:

Problem: the sum of 3 and $-5,$ increased by 18

Add Integers in Applications

In the following exercises, solve.

Exercise:

Problem:

Temperature On Monday, the high temperature in Denver was -4 degrees. Tuesday's high temperature was 20 degrees more. What was the high temperature on Tuesday?

Solution:

16 degrees

Exercise:

Problem:

Credit Frida owed $\$75$ on her credit card. Then she charged $\$21$ more. What was her new balance?

Subtract Integers

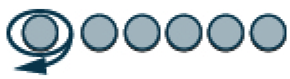
Model Subtraction of Integers

In the following exercises, model the following.

Exercise:

Problem: $6 - 1$

Solution:



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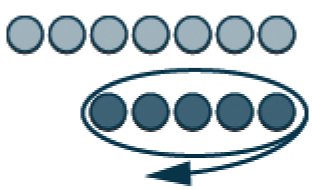
Exercise:

Problem: $-4 - (-3)$

Exercise:

Problem: $2 - (-5)$

Solution:



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Exercise:

Problem: $-1 - 4$

Simplify Expressions with Integers

In the following exercises, simplify each expression.

Exercise:

Problem: $24 - 16$

Solution:

8

Exercise:

Problem: $19 - (-9)$

Exercise:

Problem: $-31 - 7$

Solution:

−38

Exercise:

Problem: $-40 - (-11)$

Exercise:

Problem: $-52 - (-17) - 23$

Solution:

−58

Exercise:

Problem: $25 - (-3 - 9)$

Exercise:

Problem: $(1-7)-(3-8)$

Solution:

-1

Exercise:

Problem: 3^2-7^2

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

Exercise:

Problem: $x-7$ when

ⓐ $x=5$
ⓑ $x=-4$

Solution:

ⓐ -2
ⓑ -11

Exercise:

Problem: $10-y$ when

ⓐ $y=15$
ⓑ $y=-16$

Exercise:

Problem: $2n^2 - n + 5$ when $n = -4$

Solution:

41

Exercise:

Problem: $-15 - 3u^2$ when $u = -5$

Translate Phrases to Algebraic Expressions

In the following exercises, translate each phrase into an algebraic expression and then simplify.

Exercise:

Problem: the difference of -12 and 5

Solution:

$-12 - 5 = -17$

Exercise:

Problem: subtract 23 from -50

Subtract Integers in Applications

In the following exercises, solve the given applications.

Exercise:

Problem:

Temperature One morning the temperature in Bangor, Maine was 18 degrees. By afternoon, it had dropped 20 degrees. What was the afternoon temperature?

Solution:

−2 degrees

Exercise:

Problem:

Temperature On January 4, the high temperature in Laredo, Texas was 78 degrees, and the high in Houlton, Maine was -28 degrees. What was the difference in temperature of Laredo and Houlton?

Multiply and Divide Integers

Multiply Integers

In the following exercises, multiply.

Exercise:

Problem: $-9 \cdot 4$

Solution:

−36

Exercise:

Problem: $5(-7)$

Exercise:

Problem: $(-11)(-11)$

Solution:

121

Exercise:

Problem: $-1 \cdot 6$

Divide Integers

In the following exercises, divide.

Exercise:

Problem: $56 \div (-8)$

Solution:

-7

Exercise:

Problem: $-120 \div (-6)$

Exercise:

Problem: $-96 \div 12$

Solution:

-8

Exercise:

Problem: $96 \div (-16)$

Exercise:

Problem: $45 \div (-1)$

Solution:

-45

Exercise:

Problem: $-162 \div (-1)$

Simplify Expressions with Integers

In the following exercises, simplify each expression.

Exercise:

Problem: $5(-9) - 3(-12)$

Solution:

-9

Exercise:

Problem: $(-2)^5$

Exercise:

Problem: -3^4

Solution:

-81

Exercise:

Problem: $(-3)(4)(-5)(-6)$

Exercise:

Problem: $42 - 4(6 - 9)$

Solution:

54

Exercise:

Problem: $(8 - 15)(9 - 3)$

Exercise:

Problem: $-2(-18) \div 9$

Solution:

4

Exercise:

Problem: $45 \div (-3) - 12$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

Exercise:

Problem: $7x - 3$ when $x = -9$

Solution:

−66

Exercise:

Problem: $16 - 2n$ when $n = -8$

Exercise:

Problem: $5a + 8b$ when $a = -2, b = -6$

Solution:

-58

Exercise:

Problem: $x^2 + 5x + 4$ when $x = -3$

Translate Word Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

Exercise:

Problem: the product of -12 and 6

Solution:

$-12(6) = -72$

Exercise:

Problem: the quotient of 3 and the sum of -7 and s

Solve Equations using Integers; The Division Property of Equality

Determine Whether a Number is a Solution of an Equation

In the following exercises, determine whether each number is a solution of the given equation.

Exercise:

Problem: $5x - 10 = -35$

ⓐ $x = -9$
ⓑ $x = -5$
ⓒ $x = 5$

Solution:

ⓐ no
ⓑ yes
ⓒ no

Exercise:

Problem: $8u + 24 = -32$

ⓐ $u = -7$
ⓑ $u = -1$
ⓒ $u = 7$

Using the Addition and Subtraction Properties of Equality

In the following exercises, solve.

Exercise:

Problem: $a + 14 = 2$

Solution:

−12

Exercise:

Problem: $b - 9 = -15$

Solution:

−6

Exercise:

Problem: $c + (-10) = -17$

Solution:

-7

Exercise:

Problem: $d - (-6) = -26$

Solution:

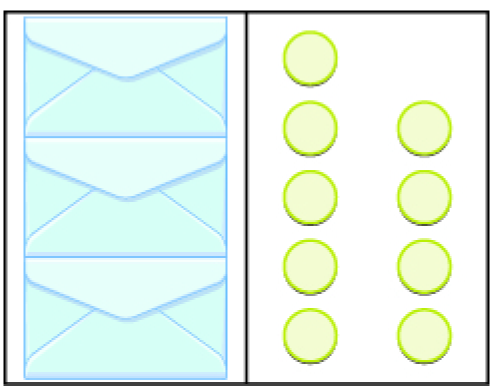
-32

Model the Division Property of Equality

In the following exercises, write the equation modeled by the envelopes and counters. Then solve it.

Exercise:

Problem:

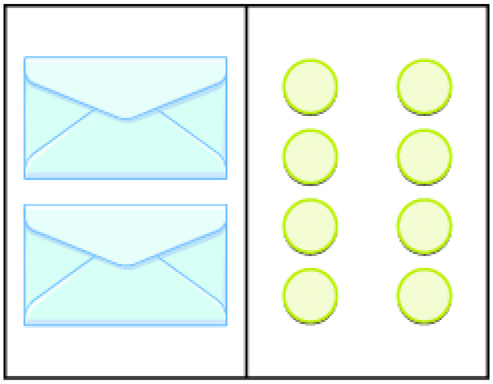


Solution:

$3x = 9$; $x = 3$;

Exercise:

Problem:



Solve Equations Using the Division Property of Equality

In the following exercises, solve each equation using the division property of equality and check the solution.

Exercise:

Problem: $8p = 72$

Solution:

9

Exercise:

Problem: $-12q = 48$

Exercise:

Problem: $-16r = -64$

Solution:

4

Exercise:

Problem: $-5s=-100$

Translate to an Equation and Solve.

In the following exercises, translate and solve.

Exercise:

Problem: The product of -6 and y is -42

Solution:

$-6y = -42$; $y = 7$

Exercise:

Problem: The difference of z and -13 is -18.

Exercise:

Problem: Four more than m is -48.

Solution:

$m + 4 = -48$; $m = -52$

Exercise:

Problem: The product of -21 and n is 63.

Everyday Math

Exercise:

Problem:

Describe how you have used two topics from this chapter in your life outside of your math class during the past month.

Chapter Practice Test

Exercise:

Problem: Locate and label $0, 2, -4$, and -1 on a number line.

In the following exercises, compare the numbers, using $<$ or $>$ or $=$.

Exercise:

Problem:

ⓐ $-6\_\_3$
ⓑ $-1\_\_-4$

Solution:

ⓐ $<$
ⓑ $>$

Exercise:

Problem:

ⓐ $-5\_\_|-5|$
ⓑ $-|-2|\_\_-2$

In the following exercises, find the opposite of each number.

Exercise:

Problem:

ⓐ -7
ⓑ 8

Solution:

ⓐ 7
ⓑ −8

In the following exercises, simplify.

Exercise:

Problem: $-(-22)$

Exercise:

Problem: $|4-9|$

Solution:

5

Exercise:

Problem: $-8+6$

Exercise:

Problem: $-15+(-12)$

Solution:

−27

Exercise:

Problem: $-7-(-3)$

Exercise:

Problem: $10 - (5 - 6)$

Solution:

11

Exercise:

Problem: $-3 \cdot 8$

Exercise:

Problem: $-6(-9)$

Solution:

54

Exercise:

Problem: $70 \div (-7)$

Exercise:

Problem: $(-2)^3$

Solution:

-8

Exercise:

Problem: -4^2

Exercise:

Problem: $16 - 3(5 - 7)$

Solution:

22

Exercise:

Problem: $|21 - 6| - |-8|$

In the following exercises, evaluate.

Exercise:

Problem: $35 - a$ when $a = -4$

Solution:

39

Exercise:

Problem: $(-2r)^2$ when $r = 3$

Exercise:

Problem: $3m - 2n$ when $m = 6,\ n = -8$

Solution:

34

Exercise:

Problem: $-|-y|$ when $y = 17$

In the following exercises, translate each phrase into an algebraic expression and then simplify, if possible.

Exercise:

Problem: the difference of −7 and −4

Solution:

−7 − (−4) = −3

Exercise:

Problem: the quotient of 25 and the sum of m and n.

In the following exercises, solve.

Exercise:

Problem:

Early one morning, the temperature in Syracuse was $-8^\circ\mathrm{F}$. By noon, it had risen 12°. What was the temperature at noon?

Solution:

4°F

Exercise:

Problem:

Collette owed $\$128$ on her credit card. Then she charged $\$65$. What was her new balance?

In the following exercises, solve.

Exercise:

Problem: $n+6=5$

Solution:

$n = -1$

Exercise:

Problem: $p - 11 = -4$

Exercise:

Problem: $-9r = -54$

Solution:

$r = 6$

In the following exercises, translate and solve.

Exercise:

Problem: The product of 15 and x is 75.

Exercise:

Problem: Eight less than y is -32.

Solution:

$y - 8 = -32$; $y = -24$

Introduction to Fractions
class="introduction"

Bakers combine ingredients to make delicious breads and pastries. (credit: Agustín Ruiz, Flickr)

Often in life, whole amounts are not exactly what we need. A baker must use a little more than a cup of milk or part of a teaspoon of sugar. Similarly a carpenter might need less than a foot of wood and a painter might use part of a gallon of paint. In this chapter, we will learn about numbers that describe parts of a whole. These numbers, called fractions, are very useful both in algebra and in everyday life. You will discover that you are already familiar with many examples of fractions!

Visualize Fractions
By the end of this section, you will be able to:

- Understand the meaning of fractions
- Model improper fractions and mixed numbers
- Convert between improper fractions and mixed numbers
- Model equivalent fractions
- Find equivalent fractions
- Simplify fractions
- Locate fractions and mixed numbers on the number line
- Order fractions and mixed numbers

Note:
Before you get started, take this readiness quiz.

1. Simplify: $5 \cdot 2 + 1$.
   If you missed this problem, review [link].
2. Divide $1{,}439 \div 4$. Check by multiplying and adding the remainder.
   If you missed this problem, review [link].
3. Fill in the blank with $<$ or $>$: $-2\_\_-5$
   If you missed this problem, review [link].

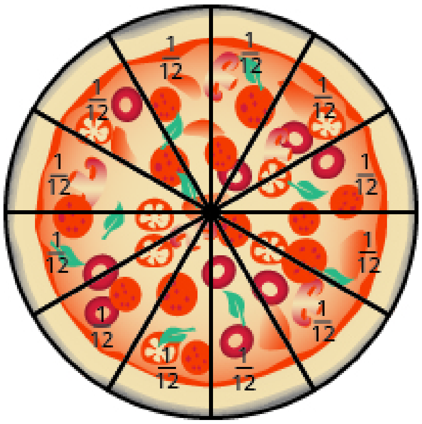
Understand the Meaning of Fractions

Andy and Bobby love pizza. On Monday night, they share a pizza equally. How much of the pizza does each one get? Are you thinking that each boy gets half of the pizza? That's right. There is one whole pizza, evenly divided into two parts, so each boy gets one of the two equal parts.

In math, we write $\frac{1}{2}$ to mean one out of two parts.

On Tuesday, Andy and Bobby share a pizza with their parents, Fred and Christy, with each person getting an equal amount of the whole pizza. How much of the pizza does each person get? There is one whole pizza, divided evenly into four equal parts. Each person has one of the four equal parts, so each has $\frac{1}{4}$ of the pizza.

On Wednesday, the family invites some friends over for a pizza dinner. There are a total of 12 people. If they share the pizza equally, each person would get $\frac{1}{12}$ of the pizza.

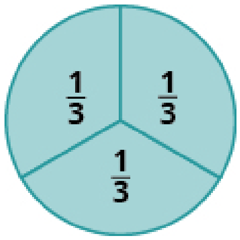


Note:
Fractions
A fraction is written $\frac{a}{b}$, where a and b are integers and $b \neq 0$. In a fraction, a is called the numerator and b is called the denominator.

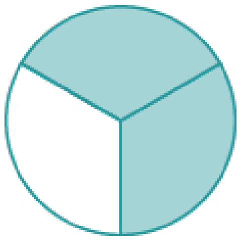
A fraction is a way to represent parts of a whole. The denominator b represents the number of equal parts the whole has been divided into, and the numerator a represents how many parts are included. The denominator, b, cannot equal zero because division by zero is undefined.

In [link], the circle has been divided into three parts of equal size. Each part represents $\frac{1}{3}$ of the circle. This type of model is called a fraction circle. Other shapes, such as rectangles, can also be used to model fractions.

Models like these are also called Area Models because the area of one unit has been divided into equal parts. As long as a known area has been divided into equal parts then we can know the area of each part without actually covering the shape with unit rectangles.



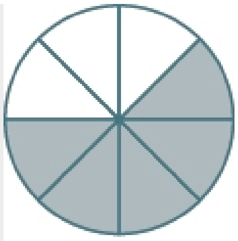
What does the fraction $\frac{2}{3}$ represent? The fraction $\frac{2}{3}$ means two of three equal parts.



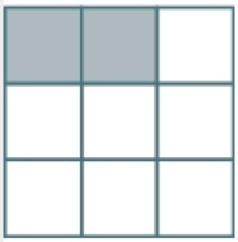
Example:
Exercise:

Problem:

Name the fraction of the shape that is shaded in each of the figures. Each of the larger figures represents 1 whole.



(a)



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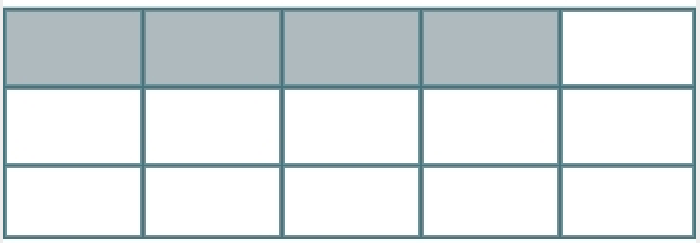


(c)

Solution:

Solution

We need to ask two questions. First, how many equal parts are there? This will be the denominator. Second, of these equal parts, how many are shaded? This will be the numerator.

ⓐ

How many equal parts are there?	There are eight equal parts.
How many are shaded?	Five parts are shaded.

Five out of eight parts are shaded. Therefore, the fraction of the circle that is shaded is $\frac{5}{8}$.

ⓑ
How many equal parts are there? There are nine equal parts.
How many are shaded? Two parts are shaded.

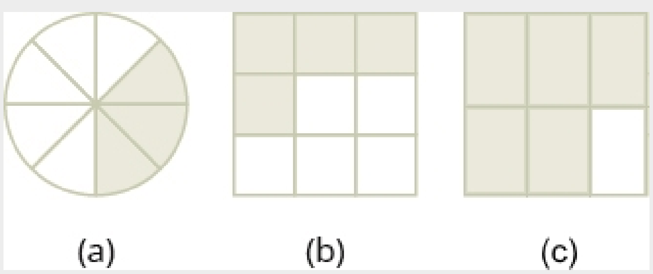
Two out of nine parts are shaded. Therefore, the fraction of the square that is shaded is $\frac{2}{9}$.

ⓒ
How many equal parts are there? There are fifteen equal parts.
How many are shaded? Four parts are shaded.

Four out of fifteen parts are shaded. Therefore, the fraction of the square that is shaded is $\frac{4}{15}$.

Note:
Exercise:

Problem: Name the fraction of the shape that is shaded in each figure:

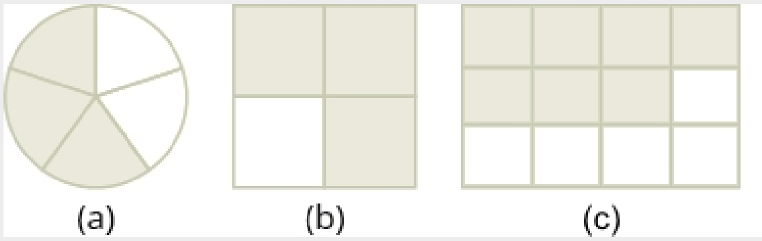


Solution:

ⓐ $\frac{3}{8}$
ⓑ $\frac{4}{9}$
ⓒ $\frac{5}{6}$

Note:
Exercise:

Problem: Name the fraction of the shape that is shaded in each figure:



Solution:

ⓐ $\frac{3}{5}$
ⓑ $\frac{3}{4}$
ⓒ $\frac{7}{12}$

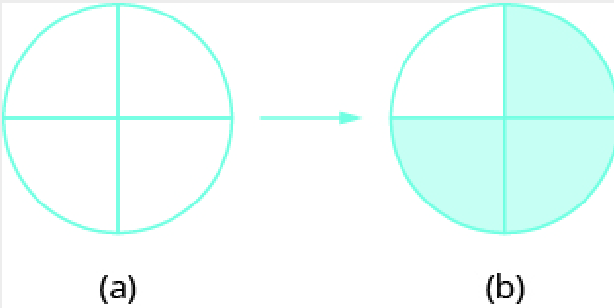
Example:
Exercise:

Problem: Shade $\frac{3}{4}$ of the circle.

Solution:
Solution

The denominator is 4, so we divide the circle into four equal parts ⓐ.

The numerator is 3, so we shade three of the four parts ⓑ.

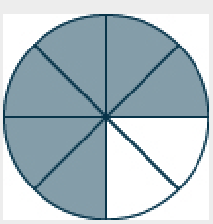


$\frac{3}{4}$ of the circle is shaded.

Note:
Exercise:

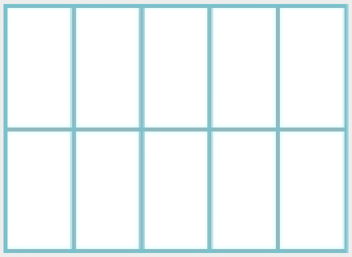
Problem: Shade $\frac{6}{8}$ of the circle.

Solution:

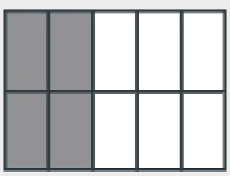


Note:
Exercise:

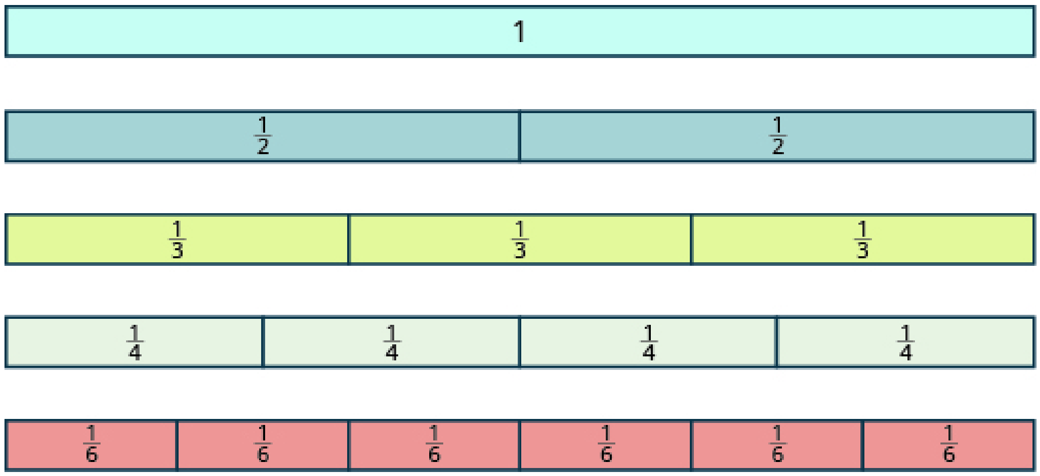
Problem: Shade $\frac{4}{10}$ of the rectangle.



Solution:



In [link] and [link], we used circles and rectangles to model fractions. Fractions can also be modeled as manipulatives called fraction tiles, as shown in [link]. Here, the whole is modeled as one long, undivided rectangular tile. Beneath it are tiles of equal length divided into different numbers of equally sized parts.



We’ll be using fraction tiles to discover some basic facts about fractions. Refer to [link] to answer the following questions:

How many $\frac{1}{2}$ tiles does it take to make one whole tile?	It takes two halves to make a whole, so two out of two is $\frac{2}{2}=1$.
How many $\frac{1}{3}$ tiles does it take to make one whole tile?	It takes three thirds, so three out of three is $\frac{3}{3}=1$.
How many $\frac{1}{4}$ tiles does it take to make one whole tile?	It takes four fourths, so four out of four is $\frac{4}{4}=1$.
How many $\frac{1}{6}$ tiles does it take to make one whole tile?	It takes six sixths, so six

	out of six is $\frac{6}{6} = 1$.
What if the whole were divided into 24 equal parts? (We have not shown fraction tiles to represent this, but try to visualize it in your mind.) How many $\frac{1}{24}$ tiles does it take to make one whole tile?	It takes 24 twenty-fourths, so $\frac{24}{24} = 1$.

It takes 24 twenty-fourths, so $\frac{24}{24} = 1$.

This leads us to the *Property of One*.

Note:
Property of One
Any number, except zero, divided by itself is one.
Equation:

$$\frac{a}{a} = 1 \qquad (a \neq 0)$$

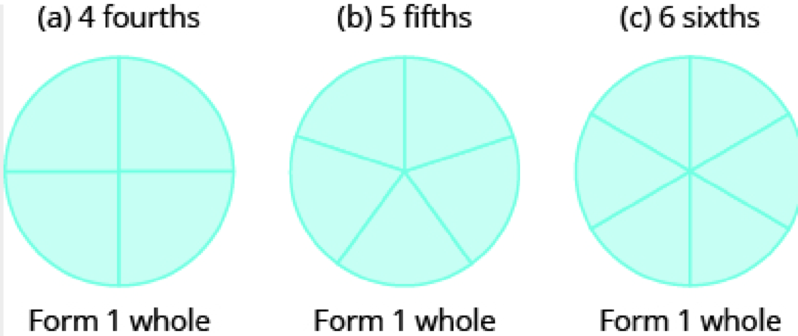
Example:
Exercise:

Problem:

Use fraction circles to make wholes using the following pieces:

ⓐ 4 fourths
ⓑ 5 fifths
ⓒ 6 sixths

Solution:
Solution

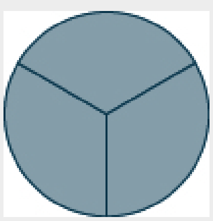


Note:
Exercise:

Problem:

Use fraction circles to make wholes with the following pieces: 3 thirds.

Solution:

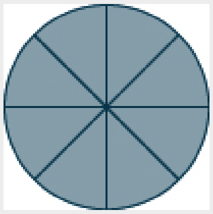


Note:
Exercise:

Problem:

Use fraction circles to make wholes with the following pieces: 8 eighths.

Solution:



What if we have more fraction pieces than we need for 1 whole? We'll look at this in the next example.

Example:
Exercise:

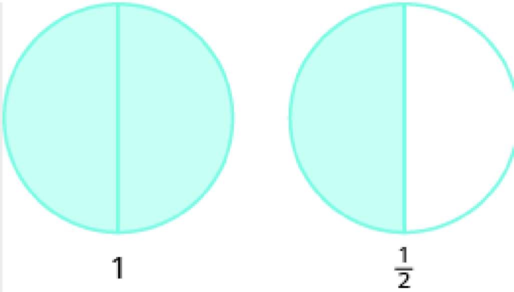
Problem:

Use fraction circles to make wholes using the following pieces:

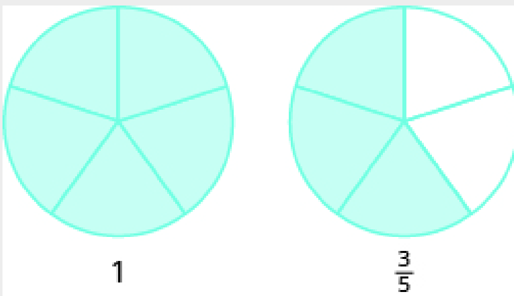
ⓐ 3 halves
ⓑ 8 fifths
ⓒ 7 thirds

Solution:
Solution

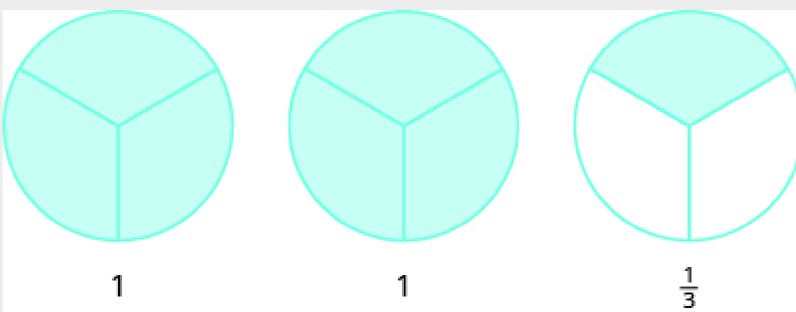
ⓐ3 halves make 1 whole with 1 half left over.



ⓑ8 fifths make 1 whole with 3 fifths left over.



ⓒ7 thirds make 2 wholes with 1 third left over.

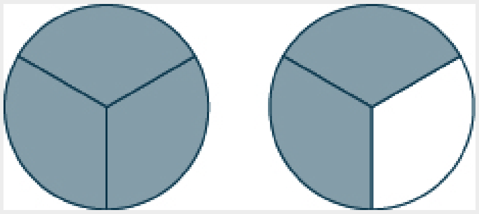


Note:
Exercise:

Problem:

Use fraction circles to make wholes with the following pieces: 5 thirds.

Solution:

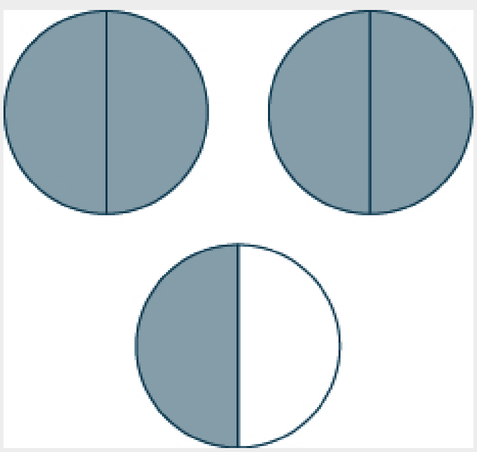


Note:
Exercise:

Problem:

Use fraction circles to make wholes with the following pieces: 5 halves.

Solution:



Model Improper Fractions and Mixed Numbers

In [link] (b), you had eight equal fifth pieces. You used five of them to make one whole, and you had three fifths left over. Let us use fraction notation to show what happened. You had eight pieces, each of them one fifth, $\frac{1}{5}$, so altogether you had eight fifths, which we can write as $\frac{8}{5}$. The fraction $\frac{8}{5}$ is one whole, 1, plus three fifths, $\frac{3}{5}$, or $1\frac{3}{5}$, which is read as *one and three-fifths*.

The number $1\frac{3}{5}$ is called a mixed number. A mixed number consists of a whole number and a fraction.

Note:
Mixed Numbers
A **mixed number** consists of a whole number a and a fraction $\frac{b}{c}$ where $c \neq 0$. It is written as follows.
Equation:

$$a\frac{b}{c} \qquad c \neq 0$$

Fractions such as $\frac{5}{4}$, $\frac{3}{2}$, $\frac{5}{5}$, and $\frac{7}{3}$ are called improper fractions. In an improper fraction, the numerator is greater than or equal to the denominator, so its value is greater than or equal to one. When a fraction has a numerator that is smaller than the denominator, it is called a proper fraction, and its value is less than one. Fractions such as $\frac{1}{2}$, $\frac{3}{7}$, and $\frac{11}{18}$ are proper fractions.

Note:
Proper and Improper Fractions
The fraction $\frac{a}{b}$ is a **proper fraction** if $a < b$ and an **improper fraction** if $a \geq b$.

Example:
Exercise:

Problem:

Name the improper fraction modeled. Then write the improper fraction as a mixed number.

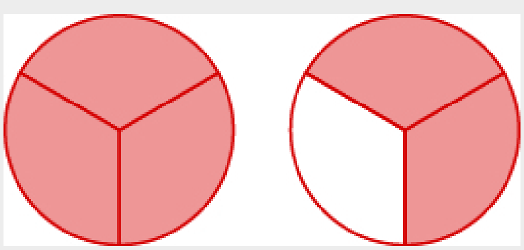
Solution:
Solution

Each circle is divided into three pieces, so each piece is $\frac{1}{3}$ of the circle. There are four pieces shaded, so there are four thirds or $\frac{4}{3}$. The figure shows that we also have one whole circle and one third, which is $1\frac{1}{3}$. So, $\frac{4}{3} = 1\frac{1}{3}$.

Note:
Exercise:

Problem:

Name the improper fraction. Then write it as a mixed number.



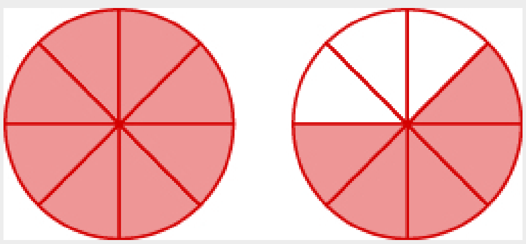
Solution:

$\frac{5}{3} = 1\frac{2}{3}$

Note:
Exercise:

Problem:

Name the improper fraction. Then write it as a mixed number.



Solution:

$\frac{13}{8} = 1\frac{5}{8}$

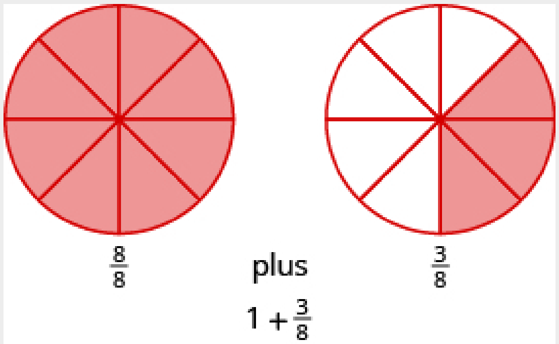
Example:
Exercise:

Problem: Draw a figure to model $\frac{11}{8}$.

Solution:
Solution

The denominator of the improper fraction is 8. Draw a circle divided into eight pieces and shade all of them. This takes care of eight eighths,

but we have 11 eighths. We must shade three of the eight parts of another circle.

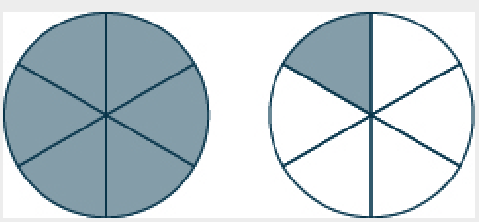


So, $\frac{11}{8} = 1\frac{3}{8}$.

Note:
Exercise:

Problem: Draw a figure to model $\frac{7}{6}$.

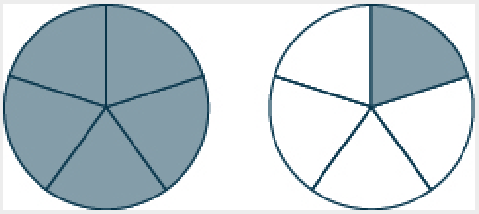
Solution:



Note:
Exercise:

Problem: Draw a figure to model $\frac{6}{5}$.

Solution:



Example:
Exercise:

Problem:

Use a model to rewrite the improper fraction $\frac{11}{6}$ as a mixed number.

Solution:
Solution

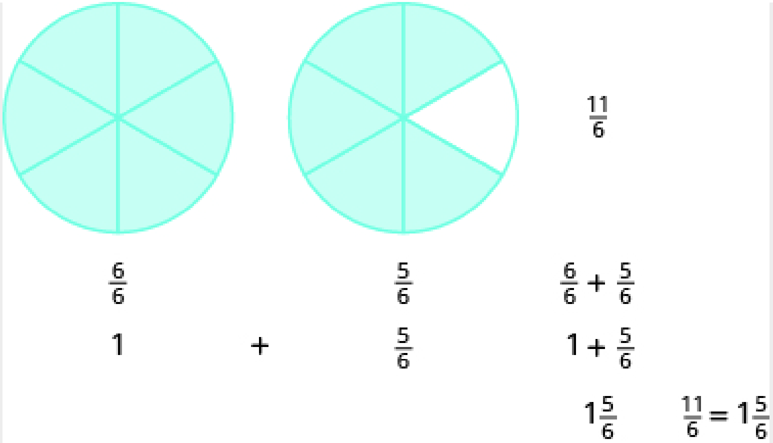
We start with 11 sixths $\left(\frac{11}{6}\right)$. We know that six sixths makes one whole.

Equation:

$$\frac{6}{6} = 1$$

That leaves us with five more sixths, which is $\frac{5}{6}$ $\left(11 \text{ sixths minus } 6 \text{ sixths is } 5 \text{ sixths}\right)$.

So, $\frac{11}{6} = 1\frac{5}{6}$.



Note:
Exercise:

Problem:

Use a model to rewrite the improper fraction as a mixed number: $\frac{9}{7}$.

Solution:

$1\frac{2}{7}$

Note:
Exercise:

Problem:

Use a model to rewrite the improper fraction as a mixed number: $\frac{7}{4}$.

Solution:

$1\frac{3}{4}$

Example:
Exercise:

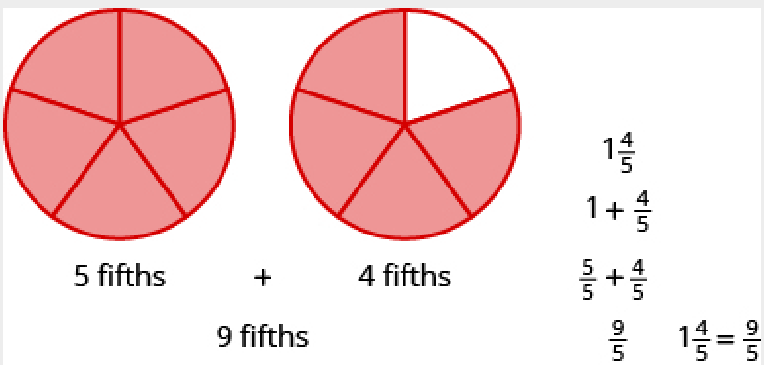
Problem:

Use a model to rewrite the mixed number $1\frac{4}{5}$ as an improper fraction.

Solution:
Solution

The mixed number $1\frac{4}{5}$ means one whole plus four fifths. The denominator is 5, so the whole is $\frac{5}{5}$. Together five fifths and four fifths equals nine fifths.

So, $1\frac{4}{5} = \frac{9}{5}$.



Note:
Exercise:

Problem:

Use a model to rewrite the mixed number as an improper fraction: $1\frac{3}{8}$.

Solution:

$\frac{11}{8}$

Note:
Exercise:

Problem:

Use a model to rewrite the mixed number as an improper fraction: $1\frac{5}{6}$.

Solution:

$\frac{11}{6}$

Convert between Improper Fractions and Mixed Numbers

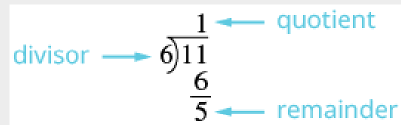
In [link], we converted the improper fraction $\frac{11}{6}$ to the mixed number $1\frac{5}{6}$ using fraction circles. We did this by grouping six sixths together to make a whole; then we looked to see how many of the 11 pieces were left. We saw that $\frac{11}{6}$ made one whole group of six sixths plus five more sixths, showing that $\frac{11}{6} = 1\frac{5}{6}$.

The division expression $\frac{11}{6}$ (which can also be written as $6\overline{)11}$) tells us to find how many groups of 6 are in 11. To convert an improper fraction to a mixed number without fraction circles, we divide.

Example:
Exercise:

Problem: Convert $\frac{11}{6}$ to a mixed number.

Solution:
Solution

	$\frac{11}{6}$
Divide the denominator into the numerator.	Remember $\frac{11}{6}$ means $11 \div 6$.
	
Identify the quotient, remainder and divisor.	
Write the mixed number as quotient $\frac{\text{remainder}}{\text{divisor}}$.	$1\frac{5}{6}$
So, $\frac{11}{6} = 1\frac{5}{6}$	

Note:
Exercise:

Problem: Convert the improper fraction to a mixed number: $\frac{13}{7}$.

Solution:

$1\frac{6}{7}$.

Note:
Exercise:

Problem: Convert the improper fraction to a mixed number: $\frac{14}{9}$.

Solution:

$1\frac{5}{9}$

Note:
Convert an improper fraction to a mixed number.

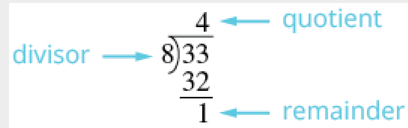
Divide the denominator into the numerator.
Identify the quotient, remainder, and divisor.
Write the mixed number as quotient $\frac{\text{remainder}}{\text{divisor}}$.

Example:
Exercise:

Problem: Convert the improper fraction $\frac{33}{8}$ to a mixed number.

Solution:
Solution

	$\frac{33}{8}$
Divide the denominator into the numerator.	Remember, $\frac{33}{8}$ means $8\overline{)33}$.

Identify the quotient, remainder, and divisor.	
Write the mixed number as quotient $\frac{\text{remainder}}{\text{divisor}}$.	$4\frac{1}{8}$
	So, $\frac{33}{8} = 4\frac{1}{8}$

Note:
Exercise:

Problem: Convert the improper fraction to a mixed number: $\frac{23}{7}$.

Solution:

$3\frac{2}{7}$

Note:
Exercise:

Problem: Convert the improper fraction to a mixed number: $\frac{48}{11}$.

Solution:

$4\frac{4}{11}$

In [link], we changed $1\frac{4}{5}$ to an improper fraction by first seeing that the whole is a set of five fifths. So we had five fifths and four more fifths.
Equation:

$$\frac{5}{5}+\frac{4}{5}=\frac{9}{5}$$

Where did the nine come from? There are nine fifths—one whole (five fifths) plus four fifths. Let us use this idea to see how to convert a mixed number to an improper fraction.

Example:
Exercise:

Problem: Convert the mixed number $4\frac{2}{3}$ to an improper fraction.

Solution:
Solution

	$4\frac{2}{3}$
Multiply the whole number by the denominator.	
The whole number is 4 and the denominator is 3.	$\frac{4\cdot 3+\square}{\square}$
Simplify.	

	$\frac{12 + \square}{\square}$
Add the numerator to the product.	
The numerator of the mixed number is 2.	$\frac{12 + 2}{\square}$
Simplify.	$\frac{14}{\square}$
Write the final sum over the original denominator.	
The denominator is 3.	$\frac{14}{3}$

Note:
Exercise:

Problem: Convert the mixed number to an improper fraction: $3\frac{5}{7}$.

Solution:

$\frac{26}{7}$

Note:
Exercise:

Problem: Convert the mixed number to an improper fraction: $2\frac{7}{8}$.

Solution:

$\frac{23}{8}$

Note:
Convert a mixed number to an improper fraction.

Multiply the whole number by the denominator.
Add the numerator to the product found in Step 1.
Write the final sum over the original denominator.

Example:
Exercise:

Problem: Convert the mixed number $10\frac{2}{7}$ to an improper fraction.

Solution:
Solution

	$10\frac{2}{7}$
Multiply the whole number by the denominator.	
The whole number is 10 and the denominator is 7.	

	$\frac{10 \cdot 7 + \square}{\square}$
Simplify.	$\frac{70 + \square}{\square}$
Add the numerator to the product.	
The numerator of the mixed number is 2.	$\frac{70 + 2}{\square}$
Simplify.	$\frac{72}{\square}$
Write the final sum over the original denominator.	
The denominator is 7.	$\frac{72}{7}$

Note:
Exercise:

Problem: Convert the mixed number to an improper fraction: $4\frac{6}{11}$.

Solution:

$\frac{50}{11}$

Note:
Exercise:

Problem: Convert the mixed number to an improper fraction: $11\frac{1}{3}$.

Solution:

$\frac{34}{3}$

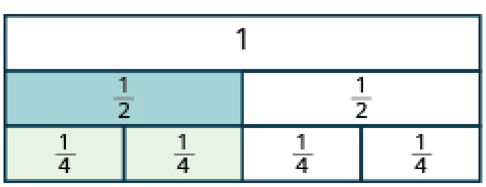
Model Equivalent Fractions

Let's think about Andy and Bobby and their favorite food again. If Andy eats $\frac{1}{2}$ of a pizza and Bobby eats $\frac{2}{4}$ of the pizza, have they eaten the same amount of pizza? In other words, does $\frac{1}{2} = \frac{2}{4}$? We can use fraction tiles to find out whether Andy and Bobby have eaten *equivalent* parts of the pizza.

Note:
Equivalent Fractions
Equivalent fractions are fractions that have the same value.

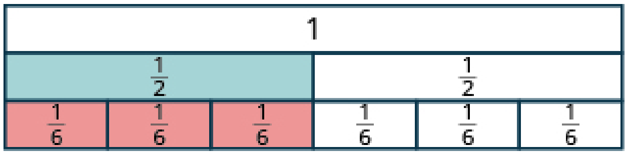
Fraction tiles serve as a useful model of equivalent fractions. You may want to use fraction tiles to do the following activity. Or you might make a copy of [link] and extend it to include eighths, tenths, and twelfths.

Start with a $\frac{1}{2}$ tile. How many fourths equal one-half? How many of the $\frac{1}{4}$ tiles exactly cover the $\frac{1}{2}$ tile?



Since two $\frac{1}{4}$ tiles cover the $\frac{1}{2}$ tile, we see that $\frac{2}{4}$ is the same as $\frac{1}{2}$, or $\frac{2}{4} = \frac{1}{2}$.

How many of the $\frac{1}{6}$ tiles cover the $\frac{1}{2}$ tile?



Since three $\frac{1}{6}$ tiles cover the $\frac{1}{2}$ tile, we see that $\frac{3}{6}$ is the same as $\frac{1}{2}$.

So, $\frac{3}{6} = \frac{1}{2}$. The fractions are equivalent fractions.

Example:
Exercise:

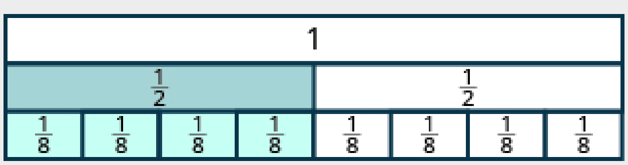
Problem:

Use fraction tiles to find equivalent fractions. Show your result with a figure.

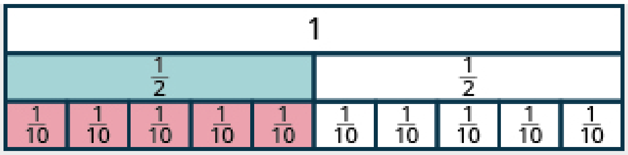
ⓐ How many eighths equal one-half?
ⓑ How many tenths equal one-half?
ⓒ How many twelfths equal one-half?

Solution:
Solution

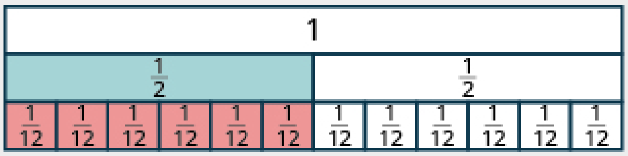
ⓐ It takes four $\frac{1}{8}$ tiles to exactly cover the $\frac{1}{2}$ tile, so $\frac{4}{8} = \frac{1}{2}$.



ⓑ It takes five $\frac{1}{10}$ tiles to exactly cover the $\frac{1}{2}$ tile, so $\frac{5}{10} = \frac{1}{2}$.



ⓒ It takes six $\frac{1}{12}$ tiles to exactly cover the $\frac{1}{2}$ tile, so $\frac{6}{12} = \frac{1}{2}$.



Suppose you had tiles marked $\frac{1}{20}$. How many of them would it take to equal $\frac{1}{2}$? Are you thinking ten tiles? If you are, you're right, because $\frac{10}{20} = \frac{1}{2}$.

We have shown that $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{6}{12}$, and $\frac{10}{20}$ are all equivalent fractions.

Note:
Exercise:

Problem:

Use fraction tiles to find equivalent fractions: How many eighths equal one-fourth?

Solution:

2

Note:
Exercise:

Problem:

Use fraction tiles to find equivalent fractions: How many twelfths equal one-fourth?

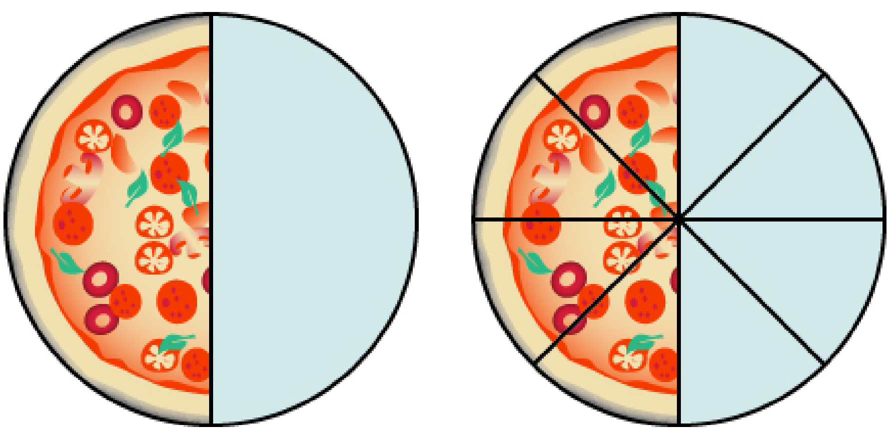
Solution:

3

Find Equivalent Fractions

We used fraction tiles to show that there are many fractions equivalent to $\frac{1}{2}$. For example, $\frac{2}{4}$, $\frac{3}{6}$, and $\frac{4}{8}$ are all equivalent to $\frac{1}{2}$. When we lined up the fraction tiles, it took four of the $\frac{1}{8}$ tiles to make the same length as a $\frac{1}{2}$ tile. This showed that $\frac{4}{8} = \frac{1}{2}$. See [link].

We can show this with pizzas, too. [link](a) shows a single pizza, cut into two equal pieces with $\frac{1}{2}$ shaded. [link](b) shows a second pizza of the same size, cut into eight pieces with $\frac{4}{8}$ shaded.



$\frac{1}{2}$
(a)

$\frac{4}{8}$
(b)

This is another way to show that $\frac{1}{2}$ is equivalent to $\frac{4}{8}$.

How can we use mathematics to change $\frac{1}{2}$ into $\frac{4}{8}$? How could you take a pizza that is cut into two pieces and cut it into eight pieces? You could cut each of the two larger pieces into four smaller pieces! The whole pizza would then be cut into eight pieces instead of just two. Mathematically, what we've described could be written as:

$$\frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$$

These models lead to the Equivalent Fractions Property, which states that if we multiply the numerator and denominator of a fraction by the same number, the value of the fraction does not change.

Note:
Equivalent Fractions Property
If a, b, and c are numbers where $b \neq 0$ and $c \neq 0$, then
Equation:

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$$

When working with fractions, it is often necessary to express the same fraction in different forms. To find equivalent forms of a fraction, we can use the Equivalent Fractions Property. For example, consider the fraction one-half.

$$\frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6} \quad \text{so} \quad \frac{1}{2} = \frac{3}{6}$$

$$\frac{1 \cdot 2}{2 \cdot 2} = \frac{2}{4} \quad \text{so} \quad \frac{1}{2} = \frac{2}{4}$$

$$\frac{1 \cdot 10}{2 \cdot 10} = \frac{10}{20} \quad \text{so} \quad \frac{1}{2} = \frac{10}{20}$$

So, we say that $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, and $\frac{10}{20}$ are equivalent fractions.

Example:
Exercise:

Problem: Find three fractions equivalent to $\frac{2}{5}$.

Solution:
Solution

To find a fraction equivalent to $\frac{2}{5}$, we multiply the numerator and denominator by the same number (but not zero). Let us multiply them by 2, 3, and 5.

$$\frac{2 \cdot 2}{5 \cdot 2} = \frac{4}{10} \qquad \frac{2 \cdot 3}{5 \cdot 3} = \frac{6}{15} \qquad \frac{2 \cdot 5}{5 \cdot 5} = \frac{10}{25}$$

So, $\frac{4}{10}$, $\frac{6}{15}$, and $\frac{10}{25}$ are equivalent to $\frac{2}{5}$.

Note:
Exercise:

Problem: Find three fractions equivalent to $\frac{3}{5}$.

Solution:

Correct answers include $\frac{6}{10}$, $\frac{9}{15}$, and $\frac{12}{20}$.

Note:
Exercise:

Problem: Find three fractions equivalent to $\frac{4}{5}$.

Solution:

Correct answers include $\frac{8}{10}$, $\frac{12}{15}$, and $\frac{16}{20}$.

Example:
Exercise:

Problem:

Find a fraction with a denominator of 21 that is equivalent to $\frac{2}{7}$.

Solution:
Solution

To find equivalent fractions, we multiply the numerator and denominator by the same number. In this case, we need to multiply the denominator by a number that will result in 21.

Since we can multiply 7 by 3 to get 21, we can find the equivalent fraction by multiplying both the numerator and denominator by 3.

$$\frac{2}{7} = \frac{2 \cdot 3}{7 \cdot 3} = \frac{6}{21}$$

Note:
Exercise:

Problem:

Find a fraction with a denominator of 21 that is equivalent to $\frac{6}{7}$.

Solution:

$\frac{18}{21}$

Note:
Exercise:

Problem:

Find a fraction with a denominator of 100 that is equivalent to $\frac{3}{10}$.

Solution:

$\frac{30}{100}$

Simplify Fractions

In working with equivalent fractions, you saw that there are many ways to write fractions that have the same value, or represent the same part of the whole. How do you know which one to use? Often, we'll use the fraction that is in *simplified* form.

A fraction is considered simplified if there are no common factors, other than 1, in the numerator and denominator. If a fraction does have common factors in the numerator and denominator, we can reduce the fraction to its simplified form by removing the common factors.

Note:
Simplified Fraction
A fraction is considered simplified if there are no common factors in the numerator and denominator.

For example,

- $\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.
- $\frac{10}{15}$ is not simplified because 5 is a common factor of 10 and 15.

The process of simplifying a fraction is often called *reducing the fraction*. In the previous section, we used the Equivalent Fractions Property to find equivalent fractions. We can also use the Equivalent Fractions Property in reverse to simplify fractions. We rewrite the property to show both forms together.

Note:
Equivalent Fractions Property
If a, b, c are numbers where $b \neq 0, c \neq 0$, then
Equation:

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c} \quad \text{and} \quad \frac{a \cdot c}{b \cdot c} = \frac{a}{b}.$$

Notice that c is a common factor in the numerator and denominator. Anytime we have a common factor in the numerator and denominator, it can be removed.

The Equivalent Fractions Property is written two ways. That was only for emphasis and was not required. A math equation can always be thought of both ways.

Note:
Simplify a fraction.

Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers.

Simplify, using the equivalent fractions property, by removing common factors.
Multiply any remaining factors.

Example:
Exercise:

Problem: Simplify: $\frac{10}{15}$.

Solution:
Solution

To simplify the fraction, we look for any common factors in the numerator and the denominator.

Notice that 5 is a factor of both 10 and 15.	$\frac{10}{15}$
Factor the numerator and denominator.	$\frac{2 \cdot 5}{3 \cdot 5}$
Remove the common factors.	$\frac{2 \cdot \cancel{5}}{3 \cdot \cancel{5}}$
Simplify.	$\frac{2}{3}$

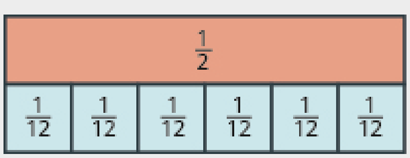
Exercise:

Problem: Simplify: $\frac{6}{12}$.

Solution:
Solution

To simplify the fraction, we again look for any common factors in the numerator and the denominator.

Notice that 2 and 3 are factors of both 6 and 12.	$\frac{6}{12}$
Factor the numerator and denominator.	$\frac{2 \cdot 3}{2 \cdot 2 \cdot 3}$
Remove the common factors. While you might be tempted to put a 0 in the numerator, that would clearly be wrong since $\frac{6}{12} \neq 0$.	$\frac{? \cdot \cancel{2} \cdot \cancel{3}}{2 \cdot \cancel{2} \cdot \cancel{3}}$
But if we write the numerator as $1 \cdot 2 \cdot 3$, then when we remove common factors we have a 1 in the numerator, and this is correct.	$\frac{1 \cdot \cancel{2} \cdot \cancel{3}}{2 \cdot \cancel{2} \cdot \cancel{3}}$
Simplify.	$\frac{1}{2}$



Note:
Exercise:

Problem: Simplify: $\frac{8}{12}$.

Solution:

$\frac{2}{3}$

Note:
Exercise:

Problem: Simplify: $\frac{12}{16}$.

Solution:

$\frac{3}{4}$

To simplify a negative fraction, we use the same process as in [link]. Remember to keep the negative sign.

Example:
Exercise:

Problem: Simplify: $-\frac{18}{24}$.

Solution:
Solution

We notice that 18 and 24 both have factors of 6.	$-\frac{18}{24}$
Rewrite the numerator and denominator showing the common factor.	$-\frac{3 \cdot 6}{4 \cdot 6}$
Remove common factors.	$-\frac{3 \cdot \cancel{6}}{4 \cdot \cancel{6}}$
Simplify.	$-\frac{3}{4}$

Note:
Exercise:

Problem: Simplify: $-\frac{21}{28}$.

Solution:

$-\frac{3}{4}$

Note:
Exercise:

Problem: Simplify: $-\frac{16}{24}$.

Solution:

$-\frac{2}{3}$

After simplifying a fraction, it is always important to check the result to make sure that the numerator and denominator do not have any more factors in common. Remember, the definition of a simplified fraction: *a fraction is considered simplified if there are no common factors in the numerator and denominator*.

When we simplify an improper fraction, there is no need to change it to a mixed number unless told to do so.

Example:
Exercise:

Problem: Simplify: $-\frac{56}{32}$.

Solution:
Solution

	$-\frac{56}{32}$
Rewrite the numerator and denominator, showing the common factors, 8.	$\frac{7 \cdot 8}{4 \cdot 8}$
Remove common factors.	$\frac{7 \cdot \cancel{8}}{4 \cdot \cancel{8}}$
Simplify.	$-\frac{7}{4}$

Note:
Exercise:

Problem: Simplify: $-\frac{54}{42}$.

Solution:

$-\frac{9}{7}$

Note:
Exercise:

Problem: Simplify: $-\frac{81}{45}$.

Solution:

$-\frac{9}{5}$

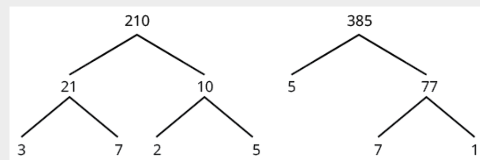
Sometimes it may not be easy to find common factors of the numerator and denominator. A good idea, then, is to factor the numerator and the denominator into prime numbers. (You may want to use the factor tree method to identify the prime factors.) Then remove the common factors using the Equivalent Fractions Property.

Example:
Exercise:

Problem: Simplify: $\frac{210}{385}$.

Solution:

Solution

Use factor trees to factor the numerator and denominator.	$\frac{210}{385}$ 
Rewrite the numerator and denominator as the product of the primes.	$\frac{210}{385} = \frac{2\cdot3\cdot5\cdot7}{5\cdot7\cdot11}$
Remove the common factors.	$\frac{2\cdot3\cdot\cancel{5}\cdot\cancel{7}}{\cancel{5}\cdot\cancel{7}\cdot11}$
Simplify.	$\frac{2\cdot3}{11}$
Multiply any remaining factors.	$\frac{6}{11}$

Note:
Exercise:

Problem: Simplify: $\frac{69}{120}$.

Solution:

$\frac{23}{40}$

Note:
Exercise:

Problem: Simplify: $\frac{120}{192}$.

Solution:

$\frac{5}{8}$

We can also simplify fractions containing variables. If a variable is a common factor in the numerator and denominator, we remove it just as we do with an integer factor.

Example:
Exercise:

Problem: Simplify: $\frac{5xy}{15x}$.

Solution:
Solution

	$\frac{5xy}{15x}$

Rewrite numerator and denominator showing common factors.	$\frac{5 \cdot x \cdot y}{3 \cdot 5 \cdot x}$
Remove common factors.	$\frac{\cancel{5} \cdot \cancel{x} \cdot y}{3 \cdot \cancel{5} \cdot \cancel{x}}$
Simplify.	$\frac{y}{3}$

Note:
Exercise:

Problem: Simplify: $\frac{7x}{7y}$.

Solution:

$\frac{x}{y}$

Note:
Exercise:

Problem: Simplify: $\frac{9a}{9b}$.

Solution:

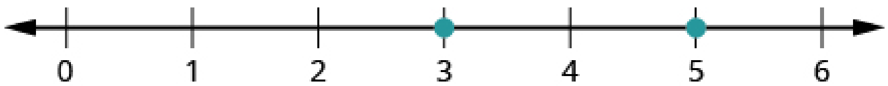
$\frac{a}{b}$

Locate Fractions and Mixed Numbers on the Number Line

Now we are ready to plot fractions on a number line. This will help us visualize fractions and understand their values.

Let us locate $\frac{1}{5}$, $\frac{4}{5}$, 3, $3\frac{1}{3}$, $\frac{7}{4}$, $\frac{9}{2}$, 5, and $\frac{8}{3}$ on the number line.

We will start with the whole numbers 3 and 5 because they are the easiest to plot.



The proper fractions listed are $\frac{1}{5}$ and $\frac{4}{5}$. We know proper fractions have values less than one, so $\frac{1}{5}$ and $\frac{4}{5}$ are located between the whole numbers 0 and 1. The denominators are both 5, so we need to divide the segment of the number line between 0 and 1 into five equal parts. We can do this by drawing four equally spaced marks on the number line, which we can then label as $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, and $\frac{4}{5}$.

Now plot points at $\frac{1}{5}$ and $\frac{4}{5}$.

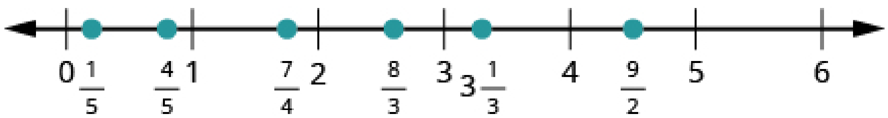
The only mixed number to plot is $3\frac{1}{3}$. Between what two whole numbers is $3\frac{1}{3}$? Remember that a mixed number is a whole number plus a proper fraction, so $3\frac{1}{3} > 3$. Since it is greater than 3, but not a whole unit greater, $3\frac{1}{3}$ is between 3 and 4. We need to divide the portion of the number line between 3 and 4 into three equal pieces (thirds) and plot $3\frac{1}{3}$ at the first mark.

Finally, look at the improper fractions $\frac{7}{4}$, $\frac{9}{2}$, and $\frac{8}{3}$. Locating these points will be easier if you change each of them to a mixed number.

Equation:

$$\frac{7}{4} = 1\frac{3}{4}, \quad \frac{9}{2} = 4\frac{1}{2}, \quad \frac{8}{3} = 2\frac{2}{3}$$

Here is the number line with all the points plotted.



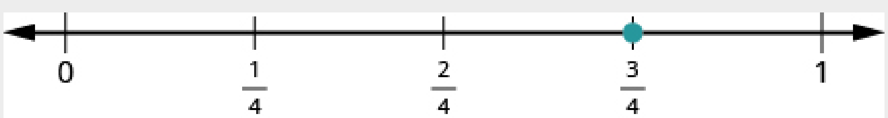
Example:
Exercise:

Problem:

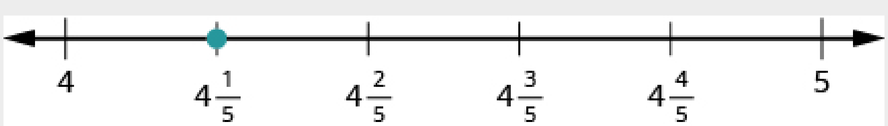
Locate and label the following on a number line: $\frac{3}{4}$, $\frac{4}{3}$, $\frac{5}{3}$, $4\frac{1}{5}$, and $\frac{7}{2}$.

Solution:
Solution

Start by locating the proper fraction $\frac{3}{4}$. It is between 0 and 1. To do this, divide the distance between 0 and 1 into four equal parts. Then plot $\frac{3}{4}$.



Next, locate the mixed number $4\frac{1}{5}$. It is between 4 and 5 on the number line. Divide the number line between 4 and 5 into five equal parts, and then plot $4\frac{1}{5}$ one-fifth of the way between 4 and 5.



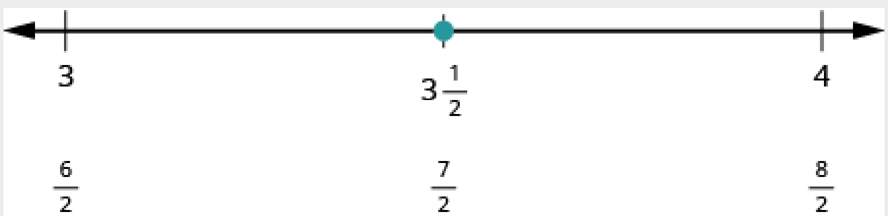
Now locate the improper fractions $\frac{4}{3}$ and $\frac{5}{3}$.

It is easier to plot them if we convert them to mixed numbers first.
Equation:

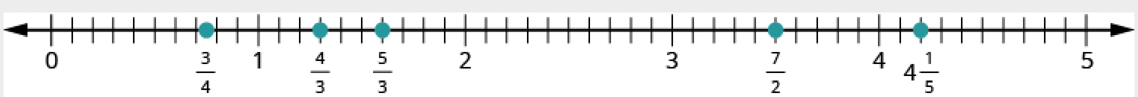
$$\frac{4}{3} = 1\frac{1}{3}, \quad \frac{5}{3} = 1\frac{2}{3}$$

Divide the distance between 1 and 2 into thirds.

Next let us plot $\frac{7}{2}$. We write it as a mixed number, $\frac{7}{2} = 3\frac{1}{2}$. Plot it between 3 and 4.



The number line shows all the numbers located on the number line.

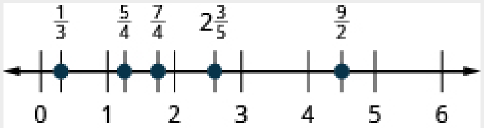


Note:
Exercise:

Problem:

Locate and label the following on a number line: $\frac{1}{3}, \frac{5}{4}, \frac{7}{4}, 2\frac{3}{5}, \frac{9}{2}$.

Solution:

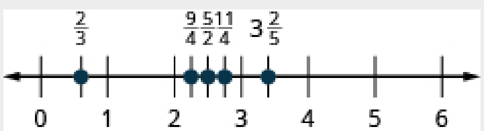


Note:
Exercise:

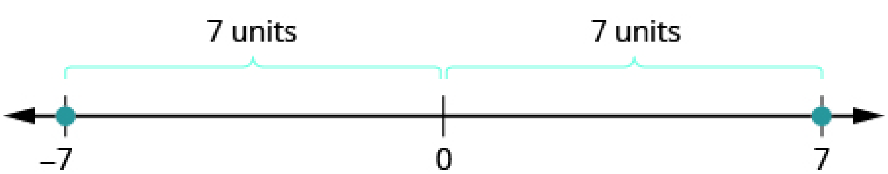
Problem:

Locate and label the following on a number line: $\frac{2}{3}$, $\frac{5}{2}$, $\frac{9}{4}$, $\frac{11}{4}$, $3\frac{2}{5}$.

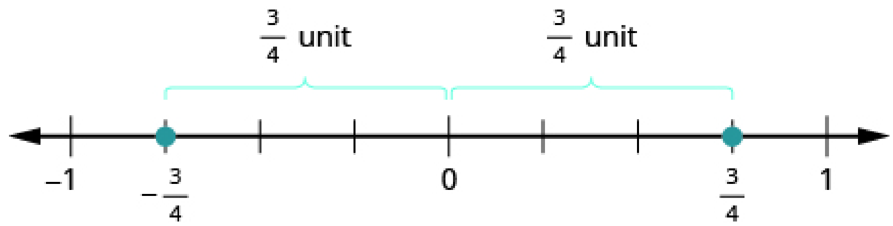
Solution:



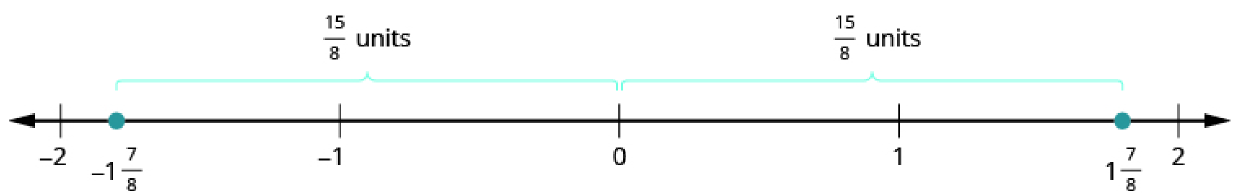
In Introduction to Integers, we defined the opposite of a number. It is the number that is the same distance from zero on the number line but on the opposite side of zero. We saw, for example, that the opposite of 7 is -7 and the opposite of -7 is 7.



Fractions have opposites, too. The opposite of $\frac{3}{4}$ is $-\frac{3}{4}$. It is the same distance from 0 on the number line, but on the opposite side of 0.



Thinking of negative fractions as the opposite of positive fractions will help us locate them on the number line. To locate $-\frac{15}{8}$ on the number line, first think of where $\frac{15}{8}$ is located. It is an improper fraction, so we first convert it to the mixed number $1\frac{7}{8}$ and see that it will be between 1 and 2 on the number line. So its opposite, $-\frac{15}{8}$, will be between -1 and -2 on the number line.



Example:
Exercise:

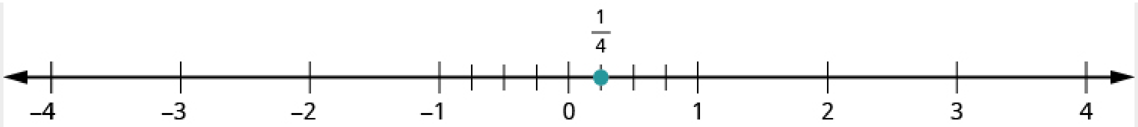
Problem:

Locate and label the following on the number line:
$\frac{1}{4}$, $-\frac{1}{4}$, $1\frac{1}{3}$, $-1\frac{1}{3}$, $\frac{5}{2}$, and $-\frac{5}{2}$.

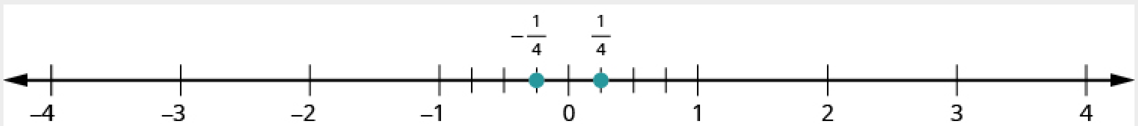
Solution:
Solution

Draw a number line. Mark 0 in the middle and then mark several units to the left and right.

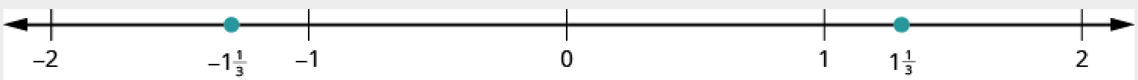
To locate $\frac{1}{4}$, divide the interval between 0 and 1 into four equal parts. Each part represents one-quarter of the distance. So plot $\frac{1}{4}$ at the first mark.



To locate $-\frac{1}{4}$, divide the interval between 0 and -1 into four equal parts. Plot $-\frac{1}{4}$ at the first mark to the left of 0.

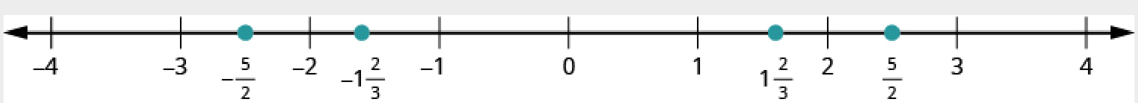


Since $1\frac{1}{3}$ is between 1 and 2, divide the interval between 1 and 2 into three equal parts. Plot $1\frac{1}{3}$ at the first mark to the right of 1. Then since $-1\frac{1}{3}$ is the opposite of $1\frac{1}{3}$ it is between -1 and -2. Divide the interval between -1 and -2 into three equal parts. Plot $-1\frac{1}{3}$ at the first mark to the left of -1.



To locate $\frac{5}{2}$ and $-\frac{5}{2}$, it may be helpful to rewrite them as the mixed numbers $2\frac{1}{2}$ and $-2\frac{1}{2}$.

Since $2\frac{1}{2}$ is between 2 and 3, divide the interval between 2 and 3 into two equal parts. Plot $\frac{5}{2}$ at the mark. Then since $-2\frac{1}{2}$ is between -2 and -3, divide the interval between -2 and -3 into two equal parts. Plot $-\frac{5}{2}$ at the mark.



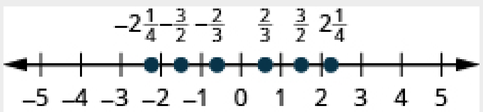
Note:
Exercise:

Problem:

Locate and label each of the given fractions on a number line:

$\frac{2}{3}, -\frac{2}{3}, 2\frac{1}{4}, -2\frac{1}{4}, \frac{3}{2}, -\frac{3}{2}$

Solution:



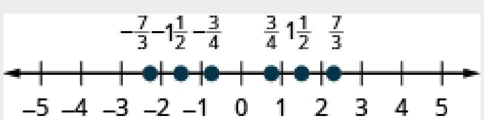
Note:
Exercise:

Problem:

Locate and label each of the given fractions on a number line:

$\frac{3}{4}, -\frac{3}{4}, 1\frac{1}{2}, -1\frac{1}{2}, \frac{7}{3}, -\frac{7}{3}$

Solution:



Order Fractions and Mixed Numbers

We can use the inequality symbols to order fractions. Remember that $a > b$ means that a is to the right of b on the number line. As we move from left to right on a number line, the values increase.

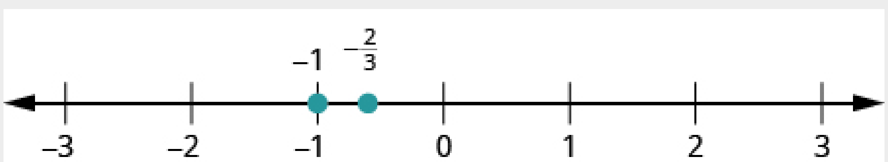
Example:
Exercise:

Problem: Order each of the following pairs of numbers, using $<$ or $>$:

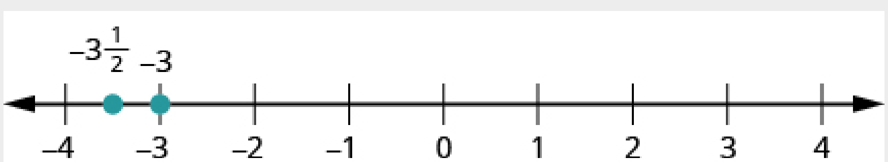
ⓐ$-\frac{2}{3}$ ____ -1
ⓑ$-3\frac{1}{2}$ ____ -3
ⓒ$-\frac{3}{7}$ ____ $-\frac{3}{8}$
ⓓ-2 ____ $\frac{-16}{9}$

Solution:
Solution

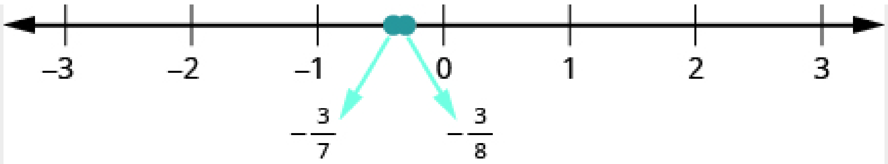
ⓐ$-\frac{2}{3} > -1$



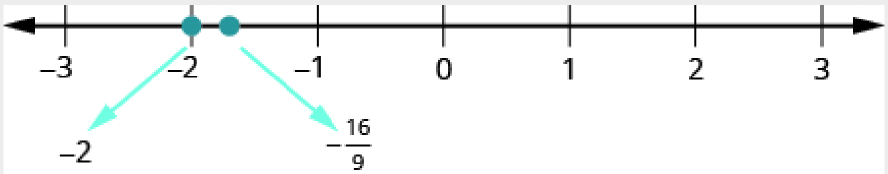
ⓑ$-3\frac{1}{2} < -3$



ⓒ$-\frac{3}{7} < -\frac{3}{8}$



ⓓ $-2 < \frac{-16}{9}$



Note:
Exercise:

Problem: Order each of the following pairs of numbers, using $<$ or $>$:

ⓐ $-\frac{1}{3}\_\_\_ -1$
ⓑ $-1\frac{1}{2}\_\_\_ -2$
ⓒ $-\frac{2}{3}\_\_\_ -\frac{1}{3}$
ⓓ $-3\_\_\_ -\frac{7}{3}$

Solution:

ⓐ $>$
ⓑ $>$
ⓒ $<$
ⓓ $<$

Note:

Exercise:

Problem: Order each of the following pairs of numbers, using $<$ or $>$:

ⓐ -3 ___ $-\frac{17}{5}$
ⓑ $-2\frac{1}{4}$ ___ -2
ⓒ $-\frac{3}{5}$ ___ $-\frac{4}{5}$
ⓓ -4 ___ $-\frac{10}{3}$

Solution:

ⓐ $>$
ⓑ $<$
ⓒ $>$
ⓓ $<$

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Introduction to Fractions
- Simplifying Fractions
- Fractions on the Number Line

Key Concepts

- **Property of One**
  - Any number, except zero, divided by itself is one.
    $\frac{a}{a} = 1$, where $a \neq 0$.
- **Mixed Numbers**

- A **mixed number** consists of a whole number a and a fraction $\frac{b}{c}$ where $c \neq 0$.
- It is written as follows: $a\frac{b}{c} \qquad c \neq 0$

- **Proper and Improper Fractions**
  - The fraction ab is a proper fraction if $a < b$ and an improper fraction if $a \geq b$.

- **Convert an improper fraction to a mixed number.**

  Divide the denominator into the numerator.
  Identify the quotient, remainder, and divisor.
  Write the mixed number as quotient $\frac{\text{remainder}}{\text{divisor}}$.

- **Convert a mixed number to an improper fraction.**

  Multiply the whole number by the denominator.
  Add the numerator to the product found in Step 1.
  Write the final sum over the original denominator.

- **Equivalent Fractions Property**
  - If a, b, and c are numbers where $b \neq 0$, $c \neq 0$, then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$.

- **Simplify a fraction.**

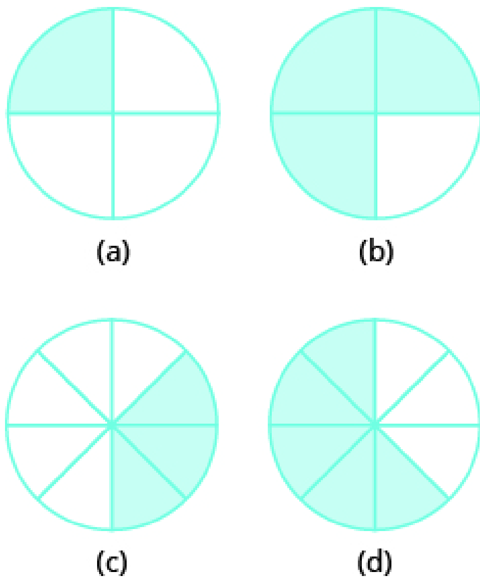
  Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers.
  Simplify, using the equivalent fractions property, by removing common factors.
  Multiply any remaining factors.

Practice Makes Perfect

In the following exercises, name the fraction of each figure that is shaded.

Exercise:

Problem:

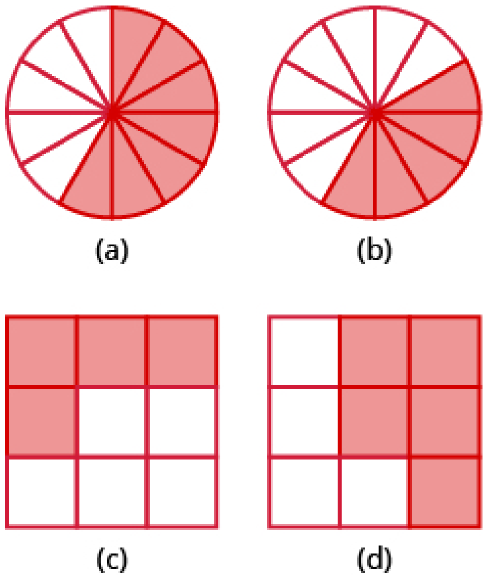


Solution:

a $\frac{1}{4}$
b $\frac{3}{4}$
c $\frac{3}{8}$
d $\frac{5}{9}$

Exercise:

Problem:

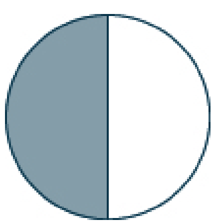


In the following exercises, shade parts of circles or squares to model the following fractions.

Exercise:

Problem: $\frac{1}{2}$

Solution:



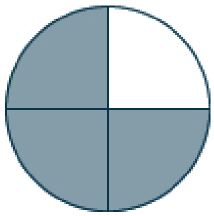
Exercise:

Problem: $\frac{1}{3}$

Exercise:

Problem: $\frac{3}{4}$

Solution:



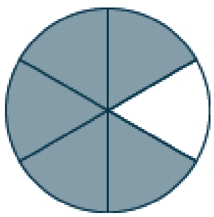
Exercise:

Problem: $\frac{2}{5}$

Exercise:

Problem: $\frac{5}{6}$

Solution:



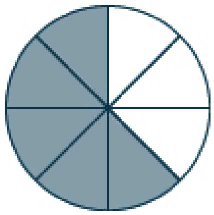
Exercise:

Problem: $\frac{7}{8}$

Exercise:

Problem: $\frac{5}{8}$

Solution:



Exercise:

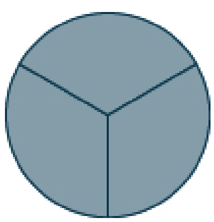
Problem: $\frac{7}{10}$

In the following exercises, use fraction circles to make wholes, if possible, with the following pieces.

Exercise:

Problem: 3 thirds

Solution:



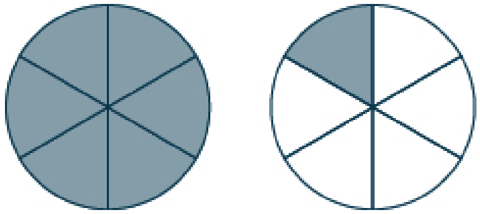
Exercise:

Problem: 8 eighths

Exercise:

Problem: 7 sixths

Solution:



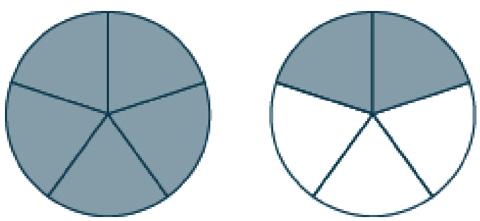
Exercise:

Problem: 4 thirds

Exercise:

Problem: 7 fifths

Solution:



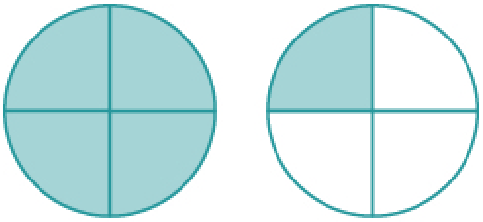
Exercise:

Problem: 7 fourths

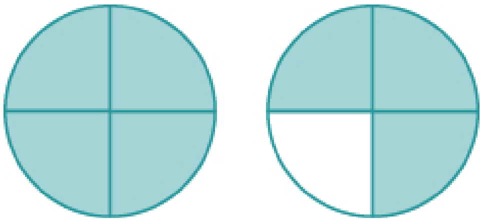
In the following exercises, name the improper fractions. Then write each improper fraction as a mixed number.

Exercise:

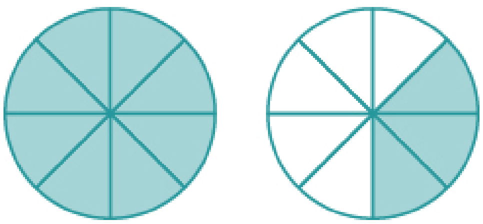
Problem:



(a)



(b)



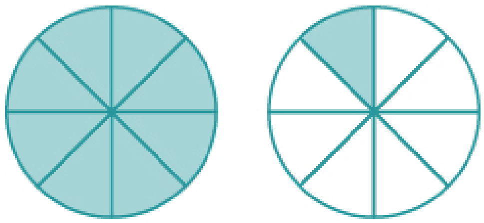
(c)

Solution:

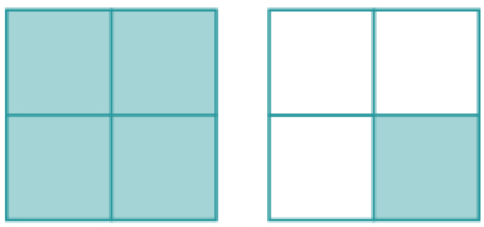
ⓐ $\frac{5}{4} = 1\frac{1}{4}$
ⓑ $\frac{7}{4} = 1\frac{3}{4}$
ⓒ $\frac{11}{8} = 1\frac{3}{8}$

Exercise:

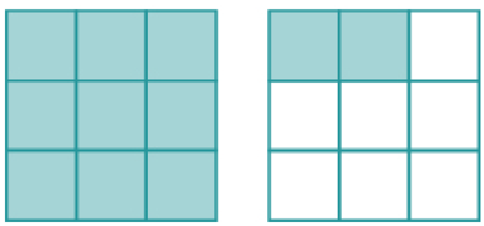
Problem:



(a)



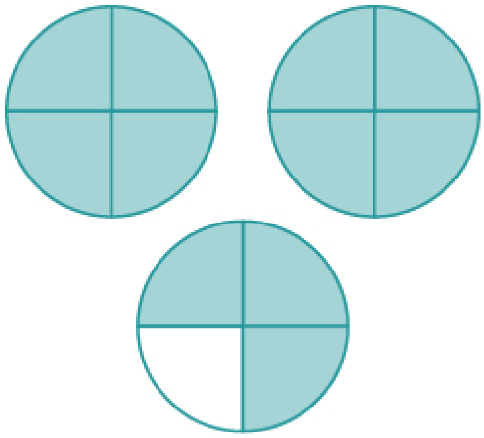
(b)



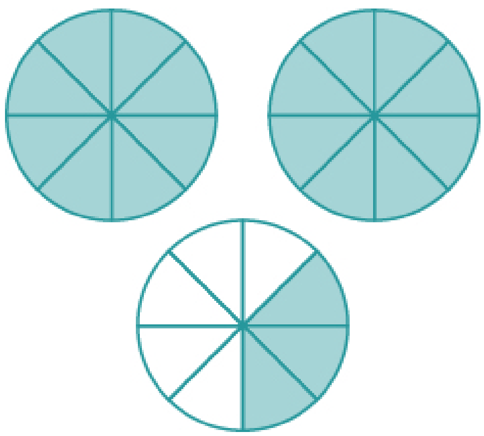
(c)

Exercise:

Problem:



(a)



(b)

Solution:

ⓐ $\frac{11}{4} = 2\frac{3}{4}$
ⓑ $\frac{19}{8} = 2\frac{3}{8}$

In the following exercises, draw fraction circles to model the given fraction.

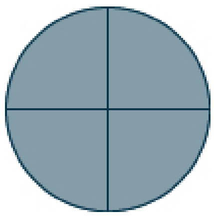
Exercise:

Problem: $\frac{3}{3}$

Exercise:

Problem: $\frac{4}{4}$

Solution:



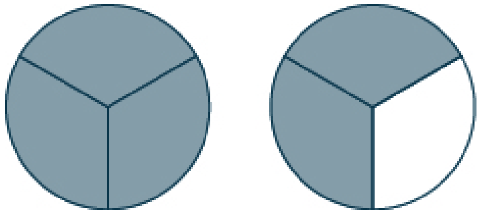
Exercise:

Problem: $\frac{7}{4}$

Exercise:

Problem: $\frac{5}{3}$

Solution:



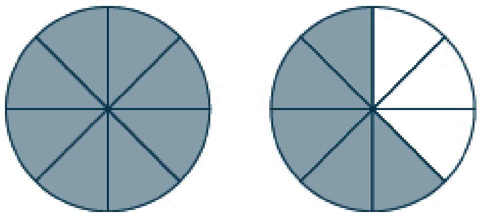
Exercise:

Problem: $\frac{11}{6}$

Exercise:

Problem: $\frac{13}{8}$

Solution:



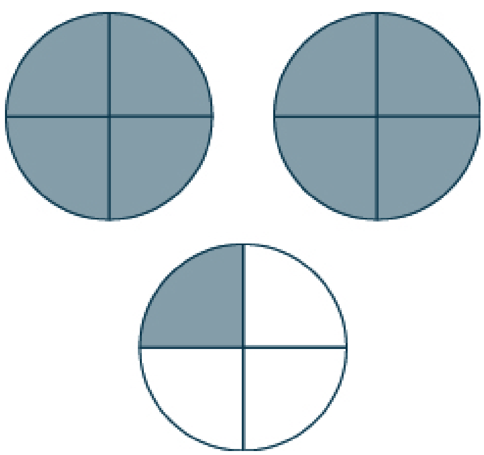
Exercise:

Problem: $\frac{10}{3}$

Exercise:

Problem: $\frac{9}{4}$

Solution:



In the following exercises, rewrite the improper fraction as a mixed number.

Exercise:

Problem: $\frac{3}{2}$

Exercise:

Problem: $\frac{5}{3}$

Solution:

$1\frac{2}{3}$

Exercise:

Problem: $\frac{11}{4}$

Exercise:

Problem: $\frac{13}{5}$

Solution:

$2\frac{3}{5}$

Exercise:

Problem: $\frac{25}{6}$

Exercise:

Problem: $\frac{28}{9}$

Solution:

$3\frac{1}{9}$

Exercise:

Problem: $\frac{42}{13}$

Exercise:

Problem: $\frac{47}{15}$

Solution:

$3\,\frac{2}{15}$

In the following exercises, rewrite the mixed number as an improper fraction.

Exercise:

Problem: $1\,\frac{2}{3}$

Exercise:

Problem: $1\,\frac{2}{5}$

Solution:

$\frac{7}{5}$

Exercise:

Problem: $2\,\frac{1}{4}$

Exercise:

Problem: $2\,\frac{5}{6}$

Solution:

$\frac{17}{6}$

Exercise:

Exercise:

Problem: $2\frac{7}{9}$

Exercise:

Problem: $2\frac{5}{7}$

Solution:

$\frac{19}{7}$

Exercise:

Problem: $3\frac{4}{7}$

Exercise:

Problem: $3\frac{5}{9}$

Solution:

$\frac{32}{9}$

In the following exercises, use fraction tiles or draw a figure to find equivalent fractions.

Exercise:

Problem: How many sixths equal one-third?

Exercise:

Problem: How many twelfths equal one-third?

Solution:

4

Exercise:

Problem: How many eighths equal three-fourths?

Exercise:

Problem: How many twelfths equal three-fourths?

Solution:

9

Exercise:

Problem: How many fourths equal three-halves?

Exercise:

Problem: How many sixths equal three-halves?

Solution:

9

In the following exercises, find three fractions equivalent to the given fraction. Show your work, using figures or algebra.

Exercise:

Problem: $\frac{1}{4}$

Exercise:

Problem: $\frac{1}{3}$

Solution:

Answers may vary. Correct answers include $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$.

Exercise:

Problem: $\frac{3}{8}$

Exercise:

Problem: $\frac{5}{6}$

Solution:

Answers may vary. Correct answers include $\frac{10}{12}, \frac{15}{18}, \frac{20}{24}$.

Exercise:

Problem: $\frac{2}{7}$

Exercise:

Problem: $\frac{5}{9}$

Solution:

Answers may vary. Correct answers include $\frac{10}{18}, \frac{15}{27}, \frac{20}{36}$.

Simplify Fractions

In the following exercises, simplify each fraction. Do not convert any improper fractions to mixed numbers.

Exercise:

Problem: $\frac{7}{21}$

Solution:

$\frac{1}{3}$

Exercise:

Problem: $\frac{8}{24}$

Exercise:

Problem: $\frac{15}{20}$

Solution:

$\frac{3}{4}$

Exercise:

Problem: $\frac{12}{18}$

Exercise:

Problem: $-\frac{40}{88}$

Solution:

$-\frac{5}{11}$

Exercise:

Problem: $-\frac{63}{99}$

Exercise:

Problem: $-\frac{108}{63}$

Solution:

$-\frac{12}{7}$

Exercise:

Problem: $-\frac{104}{48}$

Exercise:

Problem: $\frac{120}{252}$

Solution:

$\frac{10}{21}$

Exercise:

Problem: $\frac{182}{294}$

Exercise:

Problem: $-\frac{168}{192}$

Solution:

$-\frac{7}{8}$

Exercise:

Problem: $-\frac{140}{224}$

Exercise:

Problem: $\frac{11x}{11y}$

Solution:

$\frac{x}{y}$

Exercise:

Problem: $\frac{15a}{15b}$

Exercise:

Problem: $-\frac{3x}{12y}$

Solution:

$-\frac{x}{4y}$

Exercise:

Problem: $-\frac{4x}{32y}$

Exercise:

Problem: $\frac{14x^2}{21y}$

Solution:

$\frac{2x^2}{3y}$

Exercise:

Problem: $\frac{24a}{32b^2}$

In the following exercises, plot the numbers on a number line.

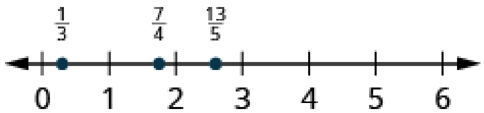
Exercise:

Problem: $\frac{2}{3}, \frac{5}{4}, \frac{12}{5}$

Exercise:

Problem: $\frac{1}{3}, \frac{7}{4}, \frac{13}{5}$

Solution:



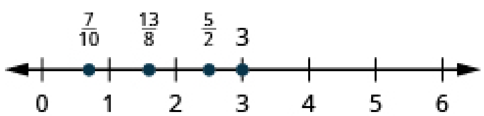
Exercise:

Problem: $\frac{1}{4}, \frac{9}{5}, \frac{11}{3}$

Exercise:

Problem: $\frac{7}{10}, \frac{5}{2}, \frac{13}{8}, 3$

Solution:



Exercise:

Problem: $2\frac{1}{3}, -2\frac{1}{3}$

Exercise:

Problem: $1\frac{3}{4}, -1\frac{3}{5}$

Solution:

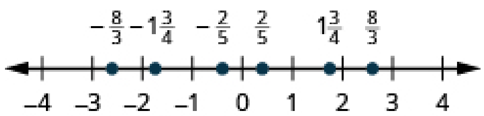
Exercise:

Problem: $\frac{3}{4}, -\frac{3}{4}, 1\frac{2}{3}, -1\frac{2}{3}, \frac{5}{2}, -\frac{5}{2}$

Exercise:

Problem: $\frac{2}{5}, -\frac{2}{5}, 1\frac{3}{4}, -1\frac{3}{4}, \frac{8}{3}, -\frac{8}{3}$

Solution:



In the following exercises, order each of the following pairs of numbers, using $<$ or $>$.

Exercise:

Problem: $-1\_\_ -\frac{1}{4}$

Exercise:

Problem: $-1\_\_ -\frac{1}{3}$

Solution:

$<$

Exercise:

Problem: $-2\frac{1}{2}\_\_ -3$

Exercise:

Problem: $-1\frac{3}{4}\_\_ -2$

Solution:

$>$

Exercise:

Problem: $-\frac{5}{12}\_\_\_ -\frac{7}{12}$

Exercise:

Problem: $-\frac{9}{10}\_\_\_ -\frac{3}{10}$

Solution:

$<$

Exercise:

Problem: $-3\_\_\_ -\frac{13}{5}$

Exercise:

Problem: $-4\_\_\_ -\frac{23}{6}$

Solution:

$<$

Everyday Math

Exercise:

Problem:

Music Measures A choreographed dance is broken into counts. A $\frac{1}{1}$ count has one step in a count, a $\frac{1}{2}$ count has two steps in a count and a $\frac{1}{3}$ count has three steps in a count. How many steps would be in a $\frac{1}{5}$ count? What type of count has four steps in it?

Exercise:

Problem:

Baking Nina is making five pans of fudge to serve after a music recital. For each pan, she needs $\frac{1}{2}$ cup of walnuts.

ⓐ How many cups of walnuts does she need for five pans of fudge?
ⓑ Do you think it is easier to measure this amount when you use an improper fraction or a mixed number? Why?

Writing Exercises

Exercise:

Problem:

Give an example from your life experience (outside of school) where it was important to understand fractions.

Solution:

Answers will vary.

Exercise:

Problem:

Rafael wanted to order half a medium pizza at a restaurant. The waiter told him that a medium pizza could be cut into 6 or 8 slices. Would he prefer 3 out of 6 slices or 4 out of 8 slices? Rafael replied that since he wasn't very hungry, he would prefer 3 out of 6 slices. Explain what is wrong with Rafael's reasoning.

Exercise:

Problem:

Explain how you locate the improper fraction $\frac{21}{4}$ on a number line on which only the whole numbers from 0 through 10 are marked.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
understand the meaning of fractions.			
model improper fractions and mixed numbers.			
convert between improper fractions and mixed numbers.			
model equivalent fractions.			
find equivalent fractions.			
locate fractions and mixed numbers on the number line.			
order fractions and mixed numbers.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

equivalent fractions
: Equivalent fractions are two or more fractions that have the same value.

fraction
: A fraction is written $\frac{a}{b}$. in a fraction, a is the numerator and b is the denominator. A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

mixed number
: A mixed number consists of a whole number a and a fraction $\frac{b}{c}$ where $c \neq 0$. It is written as $a\frac{b}{c}$, where $c \neq 0$.

proper and improper fractions
: The fraction $\frac{a}{b}$ is *proper* if $a < b$ and *improper* if $a > b$.

simplified fraction
: A fraction is considered simplified if there are no common factors in the numerator and denominator.

Multiply and Divide Fractions
By the end of this section, you will be able to:

- Multiply fractions
- Find reciprocals
- Divide fractions
- Identify and use fraction operations

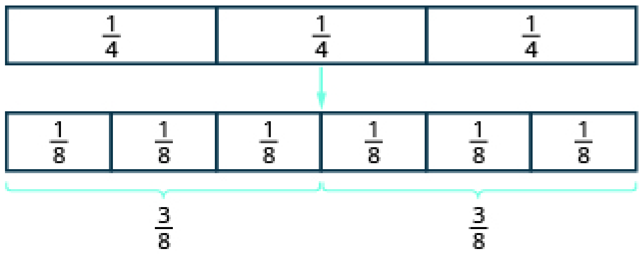
Note:
Before you get started, take this readiness quiz.

1. Find the prime factorization of 48.
   If you missed this problem, review [link].
2. Draw a model of the fraction $\frac{3}{4}$.
   If you missed this problem, review [link].
3. Find two fractions equivalent to $\frac{5}{6}$.
   Answers may vary. Acceptable answers include $\frac{10}{12}$, $\frac{15}{18}$, $\frac{50}{60}$, etc.
   If you missed this problem, review [link].

Multiply fractions

A model may help you understand multiplication of fractions. We will use fraction tiles to model $\frac{1}{2} \cdot \frac{3}{4}$. To multiply $\frac{1}{2}$ and $\frac{3}{4}$, think $\frac{1}{2}$ of $\frac{3}{4}$.

Start with fraction tiles for three-fourths. To find one-half of three-fourths, we need to divide them into two equal groups. Since we cannot divide the three $\frac{1}{4}$ tiles evenly into two parts, we exchange them for smaller tiles.

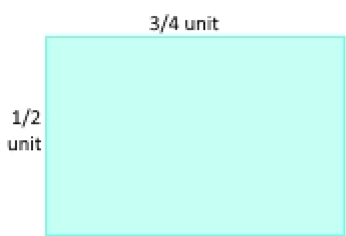


We see $\frac{6}{8}$ is equivalent to $\frac{3}{4}$. Taking half of the six $\frac{1}{8}$ tiles gives us three $\frac{1}{8}$ tiles, which is $\frac{3}{8}$.

Therefore,
Equation:

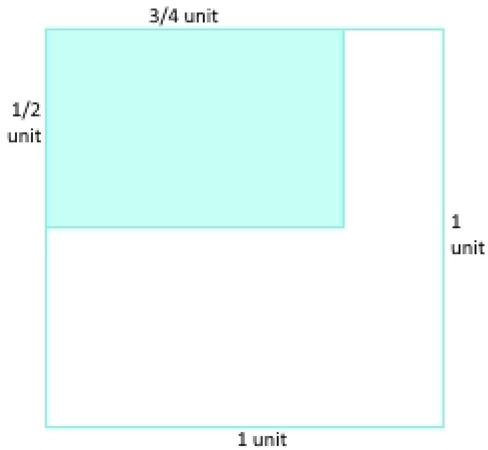
$$\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$$

The area model of multiplication can also be used to model this problem. We need a rectangle with one side 1/2 of a unit long and the other side 3/4 of a unit long.

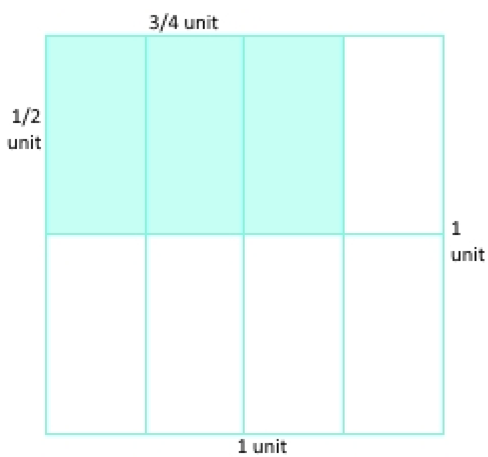


The area is the product of the sides.

We can find the area by extending the sides of this rectangle to make a unit square.



Draw in the lines that show the unit rectangle divided horizontally into half (1/2) and vertically into fourths (1/4).

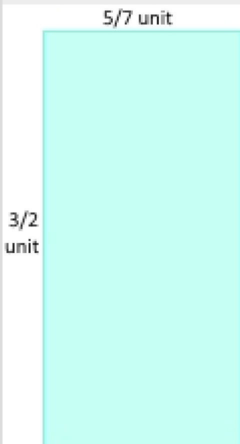


All of the small rectangles are the same size. How many of them make up the unit square?
There are 2 x 4 = 8 of them; each one is 1/8 of the unit square.

How many are shaded?
1 x 3 = 3 of them. Therefore the area is 3/8 of the square unit.

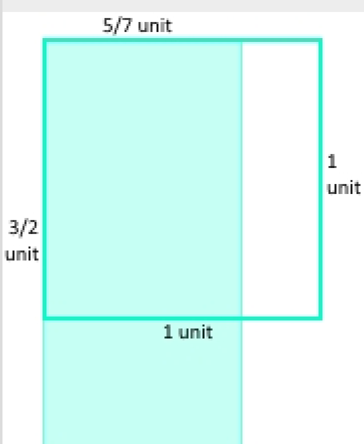
Therefore, $\frac{1}{2}$ of $\frac{3}{4}$ is $\frac{3}{8}$, or $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$.

Note:
The area model of multiplication can also be used to model $\frac{3}{2} \cdot \frac{5}{7}$.
We need a rectangle with one side 3/2 units long and the other side 5/7 of a unit long.

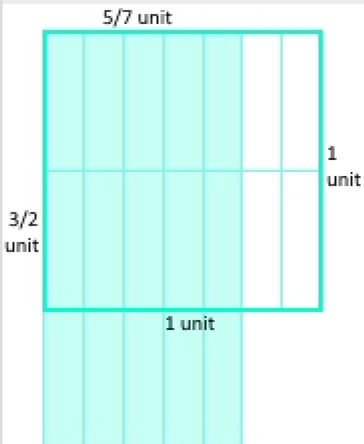


The area is the product of the sides.

We can find the area by extending the sides of this rectangle to make a unit square. The height is more than 1 unit long, so it is not fully included in the unit square.



Draw in the lines that show the unit rectangle divided horizontally into halves (1/2) and vertically into sevenths (1/7).



All of the small rectangles are the same size. How many of them make up the unit square?
There are 2 x 7 = 14 of them; each one is 1/14 of the unit square.

How many are shaded?
3 x 5 = 15 of them. Therefore the area is 15/14 of the square unit. It does not matter that some of these rectangles are outside of the unit square.

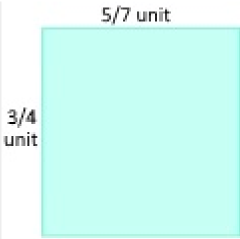
Therefore, $\frac{3}{2}$ of $\frac{5}{7}$ is $\frac{15}{14}$, or $\frac{3}{2} \cdot \frac{5}{7} = \frac{15}{14}$.

Exercise:

Problem: Use a diagram to model: $\frac{3}{4} \cdot \frac{5}{7}$.

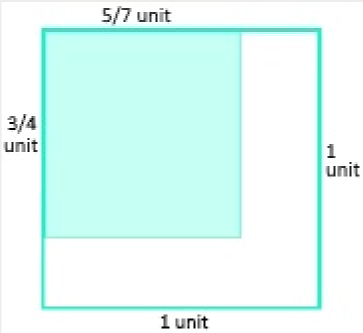
Solution:

We need a rectangle with one side 3/4 units long and the other side 5/7 of a unit long.

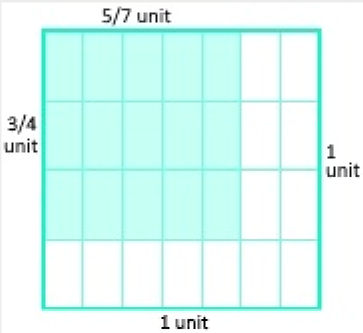


The area is the product of the sides.

We can find the area by extending the sides of this rectangle to make a unit square.



Draw in the lines that show the unit rectangle divided horizontally into quarters (1/4) and vertically into sevenths (1/7).



All of the small rectangles are the same size. How many of them make up the unit square?
There are 4 x 7 = 28 of them; each one is 1/28 of the unit square.

How many are shaded?
3 x 5 = 15 of them. Therefore the area is 15/28 of the square unit.

Therefore, $\frac{3}{4}$ of $\frac{5}{7}$ is $\frac{15}{28}$, or $\frac{3}{4} \cdot \frac{5}{7} = \frac{15}{28}$.

In the examples and exercises you may have noticed that we multiply the numerators of the factors to get the numerator of the product and multiply the denominators of the factors to get the denominator of the product. Normally, we then write the fraction in simplified form.

Note:
Fraction Multiplication
If $a, b, c,$ and d are numbers where $b \neq 0$ and $d \neq 0$, then
Equation:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

The first step in adding or subtracting fractions was to get a common denominator if the fractions didn't already have one.
That is not needed for multiplying fractions, but a common denominator still appears as the denominator of the product.
Note that bd is a multiple of both b and d.

Example:
Exercise:

Problem: Multiply, and write the answer in simplified form: $\frac{3}{4} \cdot \frac{1}{5}$.

Solution:
Solution

	$\frac{3}{4} \cdot \frac{1}{5}$
Multiply the numerators; multiply the denominators.	$\frac{3 \cdot 1}{4 \cdot 5}$
Simplify.	$\frac{3}{20}$

There are no common factors, so the fraction is simplified.

Note:
Exercise:

Problem: Multiply, and write the answer in simplified form: $\frac{1}{3} \cdot \frac{2}{5}$.

Solution:

$\frac{2}{15}$

Note:
Exercise:

Problem: Multiply, and write the answer in simplified form: $\frac{3}{5} \cdot \frac{7}{8}$.

Solution:

$\frac{21}{40}$

When multiplying fractions, the properties of positive and negative numbers still apply. It is a good idea to determine the sign of the product as the first step. In the next example we will multiply two negatives, so the product will be positive.

Example:
Exercise:

Problem:

Multiply, and write the answer in simplified form: $-\frac{5}{8}\left(-\frac{2}{3}\right)$.

Solution:
Solution

	$-\frac{5}{8}\left(-\frac{2}{3}\right)$
The signs are the same, so the product is positive. Multiply the numerators, multiply the denominators.	$\frac{5 \cdot 2}{8 \cdot 3}$

Simplify.	$\frac{10}{24}$
Look for common factors in the numerator and denominator. Rewrite showing common factors.	$\frac{5 \cdot \cancel{2}}{12 \cdot \cancel{2}}$
Remove common factors.	$\frac{5}{12}$

Another way to find this product involves removing common factors earlier.

	$-\frac{5}{8}\left(-\frac{2}{3}\right)$
Determine the sign of the product. Multiply.	$\frac{5 \cdot 2}{8 \cdot 3}$
Show common factors and then remove them.	$\frac{5 \cdot \cancel{2}}{4 \cdot \cancel{2} \cdot 3}$
Multiply remaining factors.	$\frac{5}{12}$

We get the same result.

Note:
Exercise:

Problem:

Multiply, and write the answer in simplified form: $-\frac{4}{7}\left(-\frac{5}{8}\right)$.

Solution:

$\frac{5}{14}$

Note:
Exercise:

Problem:

Multiply, and write the answer in simplified form: $-\frac{7}{12}\left(-\frac{8}{9}\right)$.

Solution:

$\frac{14}{27}$

Example:
Exercise:

Problem:

Multiply, and write the answer in simplified form: $-\frac{14}{15} \cdot \frac{20}{21}$.

Solution:
Solution

	$-\frac{14}{15}\cdot\frac{20}{21}$
Determine the sign of the product; multiply.	$-\frac{14}{15}\cdot\frac{20}{21}$
Are there any common factors in the numerator and the denominator? We know that 7 is a factor of 14 and 21, and 5 is a factor of 20 and 15.	
Rewrite showing common factors.	$-\frac{2\cdot\cancel{7}\cdot4\cdot\cancel{5}}{3\cdot\cancel{5}\cdot3\cdot\cancel{7}}$
Remove the common factors.	$-\frac{2\cdot4}{3\cdot3}$
Multiply the remaining factors.	$-\frac{8}{9}$

Note:
Exercise:

Problem:

Multiply, and write the answer in simplified form: $-\frac{10}{28}\cdot\frac{8}{15}$.

Solution:

$-\frac{4}{21}$

Note:
Exercise:

Problem:

Multiply, and write the answer in simplified form: $-\frac{9}{20} \cdot \frac{5}{12}$.

Solution:

$-\frac{3}{16}$

When multiplying a fraction by an integer, it may be helpful to write the integer as a fraction. Any integer, a, can be written as $\frac{a}{1}$. So, $3 = \frac{3}{1}$, for example.

Example:
Exercise:

Problem: Multiply, and write the answer in simplified form:

ⓐ $\frac{1}{7} \cdot 56$

ⓑ $\frac{12}{5}(-20x)$

Solution:
Solution

ⓐ	
	$\frac{1}{7} \cdot 56$

Write 56 as a fraction.	$\frac{1}{7} \cdot \frac{56}{1}$
Determine the sign of the product; multiply.	$\frac{56}{7}$
Simplify.	8

ⓑ	
	$\frac{12}{5}(-20x)$
Write −20x as a fraction.	$\frac{12}{5}\left(\frac{-20x}{1}\right)$
Determine the sign of the product; multiply.	$-\frac{12 \cdot 20 \cdot x}{5 \cdot 1}$
Show common factors and then remove them.	$-\frac{12 \cdot 4 \cdot \cancel{5}x}{\cancel{5} \cdot 1}$
Multiply remaining factors; simplify.	−48x

Note:
Exercise:

Problem: Multiply, and write the answer in simplified form:

ⓐ $\frac{1}{8} \cdot 72$
ⓑ $\frac{11}{3}(-9a)$

Solution:

ⓐ 9
ⓑ $-33a$

Note:
Exercise:

Problem: Multiply, and write the answer in simplified form:

ⓐ $\frac{3}{8} \cdot 64$
ⓑ $16x \cdot \frac{11}{12}$

Solution:

ⓐ 24
ⓑ $\frac{44x}{3}$

Find Reciprocals

The fractions $\frac{2}{3}$ and $\frac{3}{2}$ are related to each other in a special way. So are $-\frac{10}{7}$ and $-\frac{7}{10}$. Do you see how? Besides looking like upside-down versions of one another, if we were to multiply each pair of fractions, the product would be 1.

Equation:

$$\frac{2}{3} \cdot \frac{3}{2} = 1 \quad \text{and} \quad -\frac{10}{7}\left(-\frac{7}{10}\right) = 1$$

Such pairs of numbers are called reciprocals.

Note:
Reciprocal
The **reciprocal** of the fraction $\frac{a}{b}$ is $\frac{b}{a}$, where $a \neq 0$ and $b \neq 0$,
A number and its reciprocal have a product of 1. Another name for the reciprocal is **multiplicative inverse**. That is because the product or a number and its inverse result in the multiplicative identity which is 1. Notice that if X is the reciprocal of Y then Y is the reciprocal of X.
Equation:

$$\frac{a}{b} \cdot \frac{b}{a} = 1$$

To find the reciprocal of a fraction, we invert the fraction. This means that we place the numerator in the denominator and the denominator in the numerator.

To get a positive result when multiplying two numbers, the numbers must have the same sign. So reciprocals must have the same sign.

$$\frac{a}{b} \cdot \frac{b}{a} = 1 \text{ positive}$$

$$3 \cdot \frac{1}{3} = 1 \qquad \text{and} \qquad -3 \cdot \left(-\frac{1}{3}\right) = 1$$

both positive both negative

To find the reciprocal, keep the same sign and invert the fraction. The number zero does not have a reciprocal. Why? A number and its reciprocal

multiply to 1. Is there any number r so that $0 \cdot r = 1$? No. So, the number 0 does not have a reciprocal.

Example:
Exercise:

Problem:

Find the reciprocal of each number. Then check that the product of each number and its reciprocal is 1.

ⓐ $\frac{4}{9}$
ⓑ $-\frac{1}{6}$
ⓒ $-\frac{14}{5}$
ⓓ 7

Solution:
Solution

To find the reciprocals, we keep the sign and invert the fractions.

ⓐ	
Find the reciprocal of $\frac{4}{9}$.	The reciprocal of $\frac{4}{9}$ is $\frac{9}{4}$.
Check:	
Multiply the number and its reciprocal.	$\frac{4}{9} \cdot \frac{9}{4}$

Multiply numerators and denominators.	$\frac{36}{36}$
Simplify.	$1\checkmark$

ⓑ	
Find the reciprocal of $-\frac{1}{6}$.	$-\frac{6}{1}$
Simplify.	-6
Check:	$-\frac{1}{6}\cdot\left(-6\right)$
	$1\checkmark$

ⓒ	
Find the reciprocal of $-\frac{14}{5}$.	$-\frac{5}{14}$
Check:	$-\frac{14}{5}\cdot\left(-\frac{5}{14}\right)$
	$\frac{70}{70}$
	$1\checkmark$

ⓓ	
Find the reciprocal of 7.	
Write 7 as a fraction.	$\frac{7}{1}$
Write the reciprocal of $\frac{7}{1}$.	$\frac{1}{7}$
Check:	$7 \cdot \left(\frac{1}{7}\right)$
	$1 \checkmark$

Note:
Exercise:

Problem: Find the reciprocal:

ⓐ $\frac{5}{7}$
ⓑ $-\frac{1}{8}$
ⓒ $-\frac{11}{4}$
ⓓ 14

Solution:

ⓐ $\frac{7}{5}$
ⓑ -8
ⓒ $-\frac{4}{11}$
ⓓ $\frac{1}{14}$

Note:
Exercise:

Problem: Find the reciprocal:

ⓐ $\frac{3}{7}$
ⓑ $-\frac{1}{12}$
ⓒ $-\frac{14}{9}$
ⓓ 21

Solution:

ⓐ $\frac{7}{3}$
ⓑ -12
ⓒ $-\frac{9}{14}$
ⓓ $\frac{1}{21}$

In a previous chapter, we worked with opposites and absolute values. [link] compares opposites, absolute values, and reciprocals.

Opposite	Absolute Value	Reciprocal
has opposite sign	is never negative	has same sign, fraction inverts

Example:
Exercise:

Problem: Fill in the chart for each fraction in the left column:

Number	Opposite	Absolute Value	Reciprocal
$-\frac{3}{8}$			
$\frac{1}{2}$			
$\frac{9}{5}$			
-5			

Solution:
Solution

To find the opposite, change the sign. To find the absolute value, leave the positive numbers the same, but take the opposite of the negative numbers. To find the reciprocal, keep the sign the same and invert the fraction.

Number	Opposite	Absolute Value	Reciprocal

Number	Opposite	Absolute Value	Reciprocal
$-\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$-\frac{8}{3}$
$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	2
$\frac{9}{5}$	$-\frac{9}{5}$	$\frac{9}{5}$	$\frac{5}{9}$
-5	5	5	$-\frac{1}{5}$

Note:
Exercise:

Problem: Fill in the chart for each number given:

Number	Opposite	Absolute Value	Reciprocal
$-\frac{5}{8}$			
$\frac{1}{4}$			
$\frac{8}{3}$			
-8			

Solution:

Number	Opposite	Absolute Value	Reciprocal
$-\frac{5}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$-\frac{8}{5}$
$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	4
$\frac{8}{3}$	$-\frac{8}{3}$	$\frac{8}{3}$	$\frac{3}{8}$
–8	8	8	$-\frac{1}{8}$

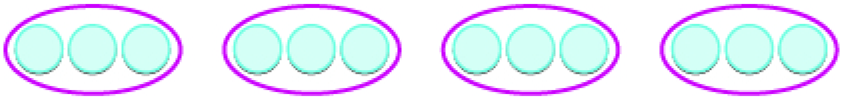
Note:
Exercise:

Problem: Fill in the chart for each number given:

Number	Opposite	Absolute Value	Reciprocal
$-\frac{4}{7}$			
$\frac{1}{8}$			
$\frac{9}{4}$			
-1			

Divide Fractions

Why is $12 \div 3 = 4$? We previously modeled this with counters. How many groups of 3 counters can be made from a group of 12 counters?



There are 4 groups of 3 counters. In other words, there are four 3s in 12.
So, $12 \div 3 = 4$.
For every division there is a related multiplication. In this case, $4 \cdot 3 = 12$.

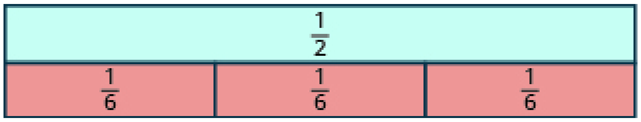
Division can be thought of as subtraction where the same number is subtracted over and over again. How many times can 3 be subtracted from 12?
$12 - 3 = 9. 9 - 3 = 6. 6 - 3 = 3. 3 - 3 = 0.$
This also gives the answer 4.

Dividing fractions is similar. Suppose we want to find the quotient: $\frac{1}{2} \div \frac{1}{6}$. We need to figure out how many $\frac{1}{6}$s there are in $\frac{1}{2}$. We can use fraction tiles to model this division. We start by lining up the half and sixth fraction tiles as shown in the next figure. Notice, there are three $\frac{1}{6}$ tiles in $\frac{1}{2}$, so $\frac{1}{2} \div \frac{1}{6} = 3$.The related multiplication problem is $3 \cdot \frac{1}{6} = \frac{1}{2}$.

Thought of as repeated subtraction, the first step is: $\frac{1}{2} - \frac{1}{6}$.
This requires a common denominator which is 6. The problem now becomes:
$\frac{3}{6} - \frac{1}{6} = \frac{2}{6}$.

$\frac{2}{6} - \frac{1}{6} = \frac{1}{6}$.

$\frac{1}{6} - \frac{1}{6} = 0$.
This also gives the answer 3.

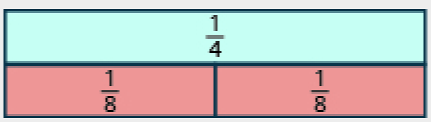


Example:
Exercise:

Problem: Model: $\frac{1}{4} \div \frac{1}{8}$. Show the corresponding multiplication.

Solution:
Solution

We want to determine how many $\frac{1}{8}$s are in $\frac{1}{4}$. Start with one $\frac{1}{4}$ tile. Line up $\frac{1}{8}$ tiles underneath the $\frac{1}{4}$ tile.



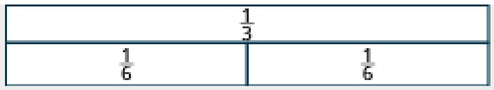
There are two $\frac{1}{8}$s in $\frac{1}{4}$.

So, $\frac{1}{4} \div \frac{1}{8} = 2$. The corresponding multiplication is $2 \cdot \frac{1}{8} = \frac{1}{4}$.

Note:
Exercise:

Problem: Model: $\frac{1}{3} \div \frac{1}{6}$. Show the corresponding multiplication.

Solution:

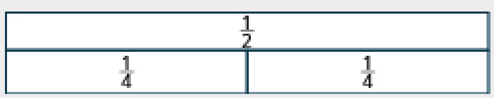


The corresponding multiplication is $2 \cdot \frac{1}{6} = \frac{1}{3}$.

Note:
Exercise:

Problem: Model: $\frac{1}{2} \div \frac{1}{4}$. Show the corresponding multiplication.

Solution:



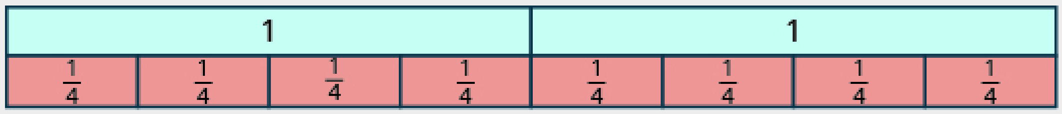
The corresponding multiplication is $2 \cdot \frac{1}{4} = \frac{1}{2}$.

Example:
Exercise:

Problem: Model: $2 \div \frac{1}{4}$. Show the corresponding multiplication.

Solution:
Solution

We are trying to determine how many $\frac{1}{4}$s there are in 2. We can model this as shown.

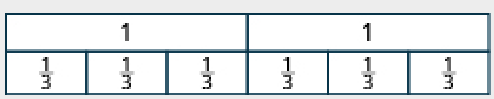


There are eight $\frac{1}{4}$s in 2.
$2 \div \frac{1}{4} = 8.$
The corresponding multiplication is $8 \cdot \frac{1}{4}$ =2.

Note:
Exercise:

Problem: Model: $2 \div \frac{1}{3}$ Show the corresponding multiplication.

Solution:

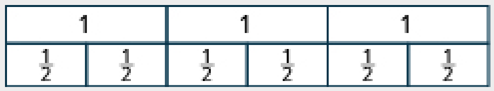


The corresponding multiplication is $6 \cdot \frac{1}{3}$ =2.

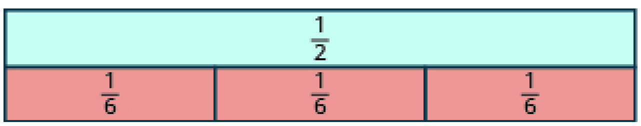
Note:
Exercise:

Problem: Model: $3 \div \frac{1}{2}$ Show the corresponding multiplication.

Solution:



The corresponding multiplication is $6 \cdot \frac{1}{2} = 3$.

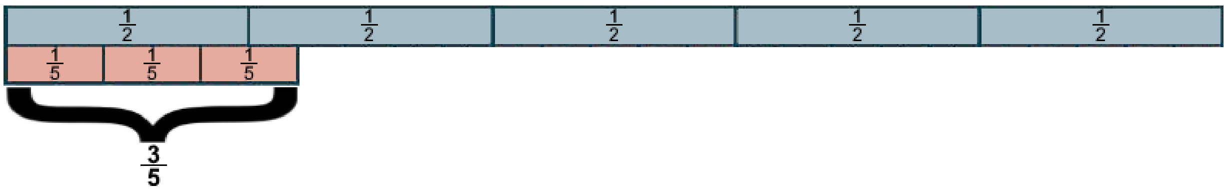


Recall $\frac{1}{2} \div \frac{1}{6}$.
Assume that we divided $\frac{1}{2}$ a foot by $\frac{1}{6}$ of a foot. The answer is still 3, not 3 feet but just the number 3. We can divide a $\frac{1}{2}$ foot long strip into 3 $\frac{1}{6}$ of a foot long strips. This is true of any other unit such as miles, pounds, or gallons. For example, if I have $\frac{1}{2}$ a pound of candy then I can divide it into 3 $\frac{1}{6}$ pound portions. The unit we use to measure doesn't matter as long as both parts are measured using the same unit.

Imagine if we measured in inches rather than in feet. 12 inches = 1 foot so $\frac{1}{2}$ foot = 6 inches and $\frac{1}{6}$ foot = 2 inches.
What is 6 inches divided by 2 inches? $6 \div 2 = 3$. This shows we can easily divide fractions if we can change the unit to get rid of the fractions. We are then left with a problem that just divides whole numbers.

The close relationship between division and subtraction means that we can use our knowledge of subtraction to help us with more difficult divisions. So far, our division of fraction problems have been easy. We've been able to illustrate them with tiles and they have worked out evenly. Let's try a harder problem: $\frac{5}{2} \div \frac{3}{5}$.

How much is left after we subtract: $\frac{5}{2} - \frac{3}{5}$?



In order to do the subtraction we need to find a common denominator. In this case, $2 \cdot 5 = 10$.

The next step is to change each fraction to an equivalent fraction with a denominator of 10.

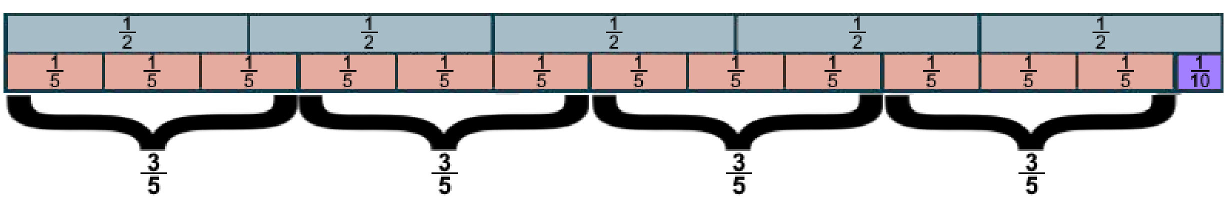
$\frac{5}{2} = \frac{5 \cdot 5}{2 \cdot 5} = \frac{25}{10}$ and $\frac{3}{5} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10}$.

Subtracting $\frac{25}{10} - \frac{6}{10} = \frac{19}{10}$.

Again $\frac{19}{10} - \frac{6}{10} = \frac{13}{10}$.

Again $\frac{13}{10} - \frac{6}{10} = \frac{7}{10}$.

Again $\frac{7}{10} - \frac{6}{10} = \frac{1}{10}$ until we can no longer take away $\frac{6}{10}$.



We were able to subtract $\frac{3}{5}$ from $\frac{5}{2}$ 4 times and $\frac{1}{10}$ was left over. The 4 is easy to understand, but what about the part that remains? It is $\frac{1}{10}$ of a unit, but what we want to know is what part of $\frac{3}{5}$ it is.

Recall that $\frac{3}{5}$ is equivalent to $\frac{6}{10}$.

Thought of that way, the individual pieces are the same size, they are both

$\frac{1}{10}$. We can easily tell that it takes $6\ \frac{1}{10}$ pieces to make $\frac{6}{10}$ so $\frac{1}{10}$ is $\frac{1}{6}$ of $\frac{6}{10}$. Therefore the final answer is $4\ \frac{1}{6}$.

Caution: $4\ \frac{1}{10}$ is not the answer. The $\frac{1}{10}$ is the size of the piece remaining, but not what part of $\frac{3}{5}$ that piece is.

Let's check our work. We check division with multiplication. For example, $12 \div 3 = 4$ is checked with $3 \cdot 4 = 12$.

Equation:

$$\frac{5}{2} \div \frac{3}{5} = 4\frac{1}{6}$$

Write as Multiplication and Change to an Improper Fraction:
Equation:

$$\frac{3}{5} \cdot 4\frac{1}{6} = \frac{3}{5} \cdot \frac{25}{6}$$

Multiplication of Fractions:
Equation:

$$\frac{3}{5} \cdot \frac{25}{6} = \frac{3 \cdot 25}{5 \cdot 6}$$

Factor:
Equation:

$$\frac{3 \cdot 25}{5 \cdot 6} = \frac{3 \cdot 5 \cdot 5}{5 \cdot 3 \cdot 2}$$

Simplify (reduce):
Equation:

$$\frac{3 \cdot 5 \cdot 5}{5 \cdot 3 \cdot 2} = \frac{5}{2}$$

If you are thinking that there must be an easier way, don't worry there is. Just like you wouldn't normally want to multiply fractions by repeated addition, you don't normally want to divide fractions by repeated subtraction. The important part is to notice that changing to a common denominator was an essential part of the solution.

Once we had changed to $\frac{25}{10}$ divided by $\frac{6}{10}$ we were almost done. Imagine that we had measured $\frac{25}{10}$ and $\frac{6}{10}$ in units that were $\frac{1}{10}$ the size. Then the amounts would have been 25 and 6 and we'd be dividing with whole numbers. Since the unit of measurement does not matter as long as it is the same for both parts, our answer would be $\frac{25}{6} = 4\frac{1}{6}$.

This agrees with the Equivalent Fraction Property.

If a, b, and c are numbers where $b \neq 0$ and $c \neq 0$, then
Equation:

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$$

c as an integer, $c \neq 0$, but it can also be a fraction. Remember that an equation can be used in either direction.

Division of Fractions with a Common Denominator

Equation:

Change to Fraction Form of Division

$$\frac{a}{c} \div \frac{b}{c} = \frac{\frac{a}{c}}{\frac{b}{c}}$$

Equation:

Multiplication of Fractions

$$\frac{\frac{a}{c}}{\frac{b}{c}} = \frac{\frac{a}{1} \cdot \frac{1}{c}}{\frac{b}{1} \cdot \frac{1}{c}}$$

Equation:

Equivalent Fraction Property

$$\frac{\frac{a}{1} \cdot \frac{1}{c}}{\frac{b}{1} \cdot \frac{1}{c}} = \frac{\frac{a}{1}}{\frac{b}{1}}$$

Equation:

Division by 1

$$\frac{\frac{a}{1}}{\frac{b}{1}} = \frac{a}{b}$$

From the example above,

Equation:

$$\frac{25}{10} \div \frac{6}{10} = \frac{\frac{25}{10}}{\frac{6}{10}}$$

Equation:

$$\frac{\frac{25}{10}}{\frac{6}{10}} = \frac{\frac{25}{1} \cdot \frac{1}{10}}{\frac{6}{1} \cdot \frac{1}{10}}$$

Equation:

$$\frac{\frac{25}{1} \cdot \frac{1}{10}}{\frac{6}{1} \cdot \frac{1}{10}} = \frac{\frac{25}{1}}{\frac{6}{1}}$$

Equation:

$$\frac{\frac{25}{1}}{\frac{6}{1}} = \frac{25}{6}$$

Dividing by 10 is the same as multiplying by $\frac{1}{10}$. In doing division of fractions, we are using the Equivalent Fraction Property going from the right side of the property to the left side. This illustrates how fraction division works. Don't go through all of the steps to get the answer. Once we have a common denominator, we can immediately write $\frac{25}{6}$.

An Intuitive Look at the Equivalent Fraction Property

How many 3 pound portions do I get when I divide 12 pounds of food? $12 \div 3 = 4$.

What if I make both the portion size and total amount of food twice as big? How many $3 \cdot 2 = 6$ pound portions do I get when I divide $12 \cdot 2 = 24$ pounds of food? $24 \div 6 = 4$.

What if I make both the portion size and total amount of food ten times as big?
How many $3 \cdot 10 = 30$ pound portions do I get when I divide $12 \cdot 10 = 120$ pounds of food? $120 \div 30 = 4$.

In each case the answer is 4 portions. Why? In each case the numerator and denominator were multiplied by the same factor. Therefore the Equivalent Fractions Property applies.

Make the portion size and the amount of food smaller by the same factor.

What if I make both the portion size and total amount of food half as big? How many $3 \div 2 = \frac{3}{2}$ pound portions do I get when I divide $12 \div 2 = \frac{12}{2}$ pounds of food? $\frac{12}{2} \div \frac{3}{2} = 4$.

What if I make both the portion size and total amount of food ten times smaller?
How many $3 \div 10 = \frac{3}{10}$ pound portions do I get when I divide $12 \div 10 = \frac{12}{10}$ pounds of food? $\frac{12}{10} \div \frac{3}{10} = 4$.

In each case the answer is 4 portions. Why? In each case, the numerator and denominator were divided by the same factor. Dividing by a factor can be thought of as multiplying by 1 divided by the same factor.

Sometimes it is easier to see the math idea when we think of money.

How many $\$3$ candies can I buy if I have 15 one dollar bills? $\$15 \div \$3 = 5$
.
How many 75 cents = 3 quarters candies can I buy if I have 15 quarters? $\frac{15}{4}$ of a dollar $\div \frac{3}{4}$ of a dollar $= 5$.
How many 30 cents = 3 dime candies can I buy if I have 15 dimes? $\frac{15}{10}$ of a dollar $\div \frac{3}{10}$ of a dollar $= 5$.

Since the denominators are the same, the division of fractions was easy. We just divided the numerators.

Dividing Fractions with Different Denominators
Just as with adding and subtracting fractions with different denominators the first step is to get a common denominator.
It is not important if the denominator is the LCM, just as long as it is a common multiple. The fraction will normally be simplified at the end of the problem.

Example:
$\frac{2}{3} \div \frac{4}{5}$

A common denominator for 3 and 5 is 15. Change each fraction to equivalent fractions with a denominator of 15.

$\frac{2 \cdot 5}{3 \cdot 5} \div \frac{4 \cdot 3}{5 \cdot 3}$

By the commutative property $3 \cdot 5 = 5 \cdot 3$. We don't have to do the extra work of seeing that is 15 (although we already did.)
Therefore the fractions have the same denominator and the problem simplifies to $\frac{2 \cdot 5}{4 \cdot 3}$. This can be simplified to $\frac{5}{6}$.

Example:
Exercise:

Problem:
Another example:

$\frac{12}{5} \div \frac{33}{42}$.

Solution:
Solution

Getting a common denominator by using the Equivalent Fractions Property twice.	$\frac{12}{5} \div \frac{33}{42} = \frac{12 \cdot 42}{5 \cdot 42} \div \frac{33 \cdot 5}{42 \cdot 5}$
Dividing fractions with a common denominator.	$\frac{12 \cdot 42}{5 \cdot 42} \div \frac{33 \cdot 5}{42 \cdot 5} = \frac{12 \cdot 42}{33 \cdot 5}$
Factoring in order to simplify.	$\frac{12 \cdot 42}{33 \cdot 5} = \frac{3 \cdot 4 \cdot 42}{3 \cdot 11 \cdot 5}$
Simplifying.	$\frac{3 \cdot 4 \cdot 42}{3 \cdot 11 \cdot 5} = \frac{4 \cdot 42}{11 \cdot 5}$
Multiplying for the final answer.	$\frac{4 \cdot 42}{11 \cdot 5} = \frac{168}{55}$.

The Algebra of Dividing Fractions

Consider two fractions $\frac{a}{b}$ and $\frac{c}{d}$ where b, c and d are not zero.

Find a common denominator for these fractions: $b \cdot d$ is a common denominator.

Change each fraction into an equivalent fraction with the common denominator.

$\frac{a}{b} = \frac{a \cdot d}{b \cdot d}$ and $\frac{c}{d} = \frac{c \cdot b}{d \cdot b}$.

$\frac{a}{b} \div \frac{c}{d} = \frac{a \cdot d}{b \cdot d} \div \frac{c \cdot b}{d \cdot b}$.

Since the denominators are equivalent, this simplifies to: $\frac{a \cdot d}{c \cdot b}$.

Note:

Dividing Fractions Using the Reciprocal

If you previously learned how to divide fractions it is probably by multiplying by the reciprocal of the divisor.

If $a, b, c,$ and d are numbers where $b \neq 0, c \neq 0,$ and $d \neq 0$, then

Equation:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

We need to say $b \neq 0, c \neq 0$ and $d \neq 0$ to be sure we don’t divide by zero.

To divide fractions, multiply the first fraction by the reciprocal of the second.

Why does this work?

Multiplying by the reciprocal is equivalent to finding a common denominator and simplifying.

Doing the multiplication: $\frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{c \cdot b}$, which is exactly the result from the common denominator method.

There is very little difference between the two methods and you can use the one you prefer now that you understand why they work.

Note:
Replacing Division with Multiplication
Negative numbers allow us to replace all subtractions with adding the opposite of the same number. This may have seemed strange at first, but it has advantages because addition is commutative and associative while subtraction is not.
Fractions enables us to replace all divisions with multiplication.
For example, $12 \div 3$ results in the same number as $12 \cdot \frac{1}{3}$. The advantage is multiplication is commutative and associative while division is not. This can save work wen simplifying an expression.
Replacing division allows for factoring and simplifying prior to multiplying.

Consider $\frac{a \cdot c \cdot b}{c} = a \cdot c \cdot b \cdot \frac{1}{c}$. By changing the division to multiplication, the expression can be simplified.
$a \cdot c \cdot b \cdot \frac{1}{c} = a \cdot b \cdot c \cdot \frac{1}{c} = a \cdot b \cdot \left(c \cdot \frac{1}{c}\right) = a \cdot b \cdot 1 = a \cdot b.$

This took a lot of steps using the associative, commutative, fractions equivalent to 1, and multiplication by 1 properties, but once you practice and think this way it can be done very quickly without writing down most of the steps. Informally, we say that the "c"s canceled. Notice that they were replaced not by a 0 but instead by a 1, which we later didn't need to write.

Replacing division with multiplication by the reciprocal does this efficiently.

Example:
Exercise:

Problem:

Divide, and write the answer in simplified form: $\frac{2}{5} \div \left(-\frac{3}{7}\right)$.

Solution:
Solution

	$\frac{2}{5} \div \left(-\frac{3}{7}\right)$
Multiply the first fraction by the reciprocal of the second.	$\frac{2}{5}\left(-\frac{7}{3}\right)$
Multiply. The product is negative.	$-\frac{14}{15}$

Note:
Exercise:

Problem:

Divide, and write the answer in simplified form: $\frac{3}{7} \div \left(-\frac{2}{3}\right)$.

Solution:

$-\frac{9}{14}$

Note:

Exercise:

Problem:

Divide, and write the answer in simplified form: $\frac{2}{3} \div \left(-\frac{7}{5}\right)$.

Solution:

$-\frac{10}{21}$

Example:
Exercise:

Problem: Divide, and write the answer in simplified form: $\frac{2}{3} \div \frac{n}{5}$.

Solution:
Solution

	$\frac{2}{3} \div \frac{n}{5}$
Multiply the first fraction by the reciprocal of the second.	$\frac{2}{3} \cdot \frac{5}{n}$
Multiply.	$\frac{10}{3n}$

Note:

Exercise:

Problem: Divide, and write the answer in simplified form: $\frac{3}{5} \div \frac{p}{7}$.

Solution:

$\frac{21}{5p}$

Note:
Exercise:

Problem: Divide, and write the answer in simplified form: $\frac{5}{8} \div \frac{q}{3}$.

Solution:

$\frac{15}{8q}$

Example:
Exercise:

Problem:

Divide, and write the answer in simplified form: $-\frac{3}{4} \div \left(-\frac{7}{8}\right)$.

Solution:
Solution

	$-\frac{3}{4} \div \left(-\frac{7}{8}\right)$
Multiply the first fraction by the reciprocal of the second.	$-\frac{3}{4} \cdot \left(-\frac{8}{7}\right)$
Multiply. Remember to determine the sign first.	$\frac{3 \cdot 8}{4 \cdot 7}$
Rewrite to show common factors.	$\frac{3 \cdot \cancel{4} \cdot 2}{\cancel{4} \cdot 7}$
Remove common factors and simplify.	$\frac{6}{7}$

Note:
Exercise:

Problem:

Divide, and write the answer in simplified form: $-\frac{2}{3} \div \left(-\frac{5}{6}\right)$.

Solution:

$\frac{4}{5}$

Note:
Exercise:

Problem:

Divide, and write the answer in simplified form: $-\frac{5}{6} \div \left(-\frac{2}{3}\right)$.

Solution:

$\frac{5}{4}$

Example:
Exercise:

Problem: Divide, and write the answer in simplified form: $\frac{7}{18} \div \frac{14}{27}$.

Solution:
Solution

	$\frac{7}{18} \div \frac{14}{27}$
Multiply the first fraction by the reciprocal of the second.	$\frac{7}{18} \cdot \frac{27}{14}$
Multiply.	$\frac{7 \cdot 27}{18 \cdot 14}$
Rewrite showing common factors.	$\frac{\cancel{7} \cdot \cancel{9} \cdot 3}{\cancel{9} \cdot 2 \cdot \cancel{7} \cdot 2}$
Remove common factors.	$\frac{3}{2 \cdot 2}$
Simplify.	$\frac{3}{4}$

Note:
Exercise:

Problem: Divide, and write the answer in simplified form: $\frac{7}{27} \div \frac{35}{36}$.

Solution:

$\frac{4}{15}$

Note:
Exercise:

Problem: Divide, and write the answer in simplified form: $\frac{5}{14} \div \frac{15}{28}$.

Solution:

$\frac{2}{3}$

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Multiplying Fractions (Positive Only)
- Multiplying Signed Fractions
- Dividing Fractions (Positive Only)
- Dividing Signed Fractions

Identify and Use Fraction Operations

By now in this chapter, you have practiced multiplying, dividing, adding, and subtracting fractions. The following table summarizes these four fraction operations. Remember: You need a common denominator to add or subtract fractions, but not to multiply or divide fractions

Note:
Summary of Fraction Operations
Fraction multiplication: Multiply the numerators and multiply the denominators.
Equation:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Fraction division: Multiply the first fraction by the reciprocal of the second.
Equation:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

Fraction addition: Add the numerators and place the sum over the common denominator. If the fractions have different denominators, first convert them to equivalent forms with the LCD.
Equation:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

Fraction subtraction: Subtract the numerators and place the difference over the common denominator. If the fractions have different denominators, first convert them to equivalent forms with the LCD.
Equation:

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

Example:
Exercise:

Problem: Simplify:

ⓐ $-\frac{1}{4}+\frac{1}{6}$
ⓑ $-\frac{1}{4}\div\frac{1}{6}$

Solution:
Solution

First we ask ourselves, “What is the operation?”

ⓐThe operation is addition.

Do the fractions have a common denominator? No.

	$-\frac{1}{4}+\frac{1}{6}$
Find the LCD. $$\begin{aligned} 4 &= 2\cdot 2 \\ 6 &= 2\cdot \quad 3 \\ \hline \text{LCD} &= 2\cdot 2\cdot 3 \\ \text{LCD} &= 12 \end{aligned}$$	
Rewrite each fraction as an equivalent fraction with the LCD.	$-\frac{1\cdot 3}{4\cdot 3}+\frac{1\cdot 2}{6\cdot 2}$

Simplify the numerators and denominators.	$-\frac{3}{12}+\frac{2}{12}$
Add the numerators and place the sum over the common denominator.	$-\frac{1}{12}$
Check to see if the answer can be simplified. It cannot.	

ⓑThe operation is division. We do not need a common denominator.

	$-\frac{1}{4}\div\frac{1}{6}$
To divide fractions, multiply the first fraction by the reciprocal of the second.	$-\frac{1}{4}\cdot\frac{6}{1}$
Multiply.	$-\frac{6}{4}$
Simplify.	$-\frac{3}{2}$

Note:
Exercise:

Problem: Simplify each expression:

ⓐ$-\frac{3}{4}-\frac{1}{6}$
ⓑ$-\frac{3}{4}\cdot\frac{1}{6}$

Solution:

ⓐ $-\frac{11}{12}$
ⓑ $-\frac{1}{8}$

Note:
Exercise:

Problem: Simplify each expression:

ⓐ $\frac{5}{6} \div \left(-\frac{1}{4}\right)$
ⓑ $\frac{5}{6} - \left(-\frac{1}{4}\right)$

Solution:

ⓐ $-\frac{10}{3}$
ⓑ $\frac{13}{12}$

Example:
Exercise:

Problem: Simplify:

ⓐ mfrac $- \frac{3}{10}$
ⓑ mfrac $\cdot \frac{3}{10}$

Solution:

Solution

ⓐ The operation is subtraction. The fractions do not have a common denominator.

	$\frac{5x}{6} - \frac{3}{10}$
Rewrite each fraction as an equivalent fraction with the LCD, 30.	$\frac{5x\cdot5}{6\cdot5} - \frac{3\cdot3}{10\cdot3}$
	$\frac{25x}{30} - \frac{9}{30}$
Subtract the numerators and place the difference over the common denominator.	$\frac{25x-9}{30}$

ⓑ The operation is multiplication; no need for a common denominator.

	$\frac{5x}{6} \cdot \frac{3}{10}$
To multiply fractions, multiply the numerators and multiply the denominators.	$\frac{5x\cdot3}{6\cdot10}$
Rewrite, showing common factors.	$\frac{\cancel{5}\cdot x\cdot\cancel{3}}{2\cdot\cancel{3}\cdot2\cdot\cancel{5}}$
Remove common factors to simplify.	$\frac{x}{4}$

Note:
Exercise:

Problem: Simplify:

ⓐ $\frac{3a}{4} - \frac{8}{9}$
ⓑ $\frac{3a}{4} \cdot \frac{8}{9}$

Solution:

ⓐmfrac
ⓑmfrac

Note:
Exercise:

Problem: Simplify:

ⓐ $\frac{4k}{5} + \frac{5}{6}$
ⓑ $\frac{4k}{5} \div \frac{5}{6}$

Solution:

ⓐmfrac
ⓑmfrac

Key Concepts

- **Fraction Multiplication**
  - If $a, b, c,$ and d are numbers where $b \neq 0$and $d \neq 0$, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.
- **Reciprocal**
  - A number and its reciprocal have a product of 1. $\frac{a}{b} \cdot \frac{b}{a} = 1$
  -

Opposite	Absolute Value	Reciprocal
has opposite sign	is never negative	has same sign, fraction inverts

- **Fraction Division**
  - If $a, b, c,$ and d are numbers where $b \neq 0$, $c \neq 0$ and $d \neq 0$, then **Equation:**

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

  - To divide fractions, multiply the first fraction by the reciprocal of the second.

Practice Makes Perfect

Simplify Fractions

In the following exercises, simplify each fraction. Do not convert any improper fractions to mixed numbers.

Exercise:

Problem: $\frac{7}{21}$

Solution:

$\frac{1}{3}$

Exercise:

Problem: $\frac{8}{24}$

Exercise:

Problem: $\frac{15}{20}$

Solution:

$\frac{3}{4}$

Exercise:

Problem: $\frac{12}{18}$

Exercise:

Problem: $-\frac{40}{88}$

Solution:

$-\frac{5}{11}$

Exercise:

Problem: $-\frac{63}{99}$

Exercise:

Problem: $-\frac{108}{63}$

Solution:

$-\frac{12}{7}$

Exercise:

Problem: $-\frac{104}{48}$

Exercise:

Problem: $\frac{120}{252}$

Solution:

$\frac{10}{21}$

Exercise:

Problem: $\frac{182}{294}$

Exercise:

Problem: $-\frac{168}{192}$

Solution:

$-\frac{7}{8}$

Exercise:

Problem: $-\frac{140}{224}$

Exercise:

Problem: $\frac{11x}{11y}$

Solution:

$\frac{x}{y}$

Exercise:

Problem: $\frac{15a}{15b}$

Exercise:

Problem: $-\frac{3x}{12y}$

Solution:

$-\frac{x}{4y}$

Exercise:

Problem: $-\frac{4x}{32y}$

Exercise:

Problem: $\frac{14x^2}{21y}$

Solution:

$\frac{2x^2}{3y}$

Exercise:

Problem: $\frac{24a}{32b^2}$

Multiply Fractions

In the following exercises, use a diagram to model.

Exercise:

Problem: $\frac{1}{2} \cdot \frac{2}{3}$

Solution:

$\frac{1}{3}$

Exercise:

Problem: $\frac{1}{2} \cdot \frac{5}{8}$

Exercise:

Problem: $\frac{1}{3} \cdot \frac{5}{6}$

Solution:

$\frac{5}{18}$

Exercise:

Problem: $\frac{1}{3} \cdot \frac{2}{5}$

In the following exercises, multiply, and write the answer in simplified form.

Exercise:

Problem: $\frac{2}{5} \cdot \frac{1}{3}$

Solution:

$\frac{2}{15}$

Exercise:

Problem: $\frac{1}{2} \cdot \frac{3}{8}$

Exercise:

Problem: $\frac{3}{4} \cdot \frac{9}{10}$

Solution:

$\frac{27}{40}$

Exercise:

Problem: $\frac{4}{5} \cdot \frac{2}{7}$

Exercise:

Problem: $-\frac{2}{3}\left(-\frac{3}{8}\right)$

Solution:

$\frac{1}{4}$

Exercise:

Problem: $-\frac{3}{4}\left(-\frac{4}{9}\right)$

Exercise:

Problem: $-\frac{5}{9} \cdot \frac{3}{10}$

Solution:

$-\frac{1}{6}$

Exercise:

Problem: $-\frac{3}{8} \cdot \frac{4}{15}$

Exercise:

Problem: $\frac{7}{12}\left(-\frac{8}{21}\right)$

Solution:

$-\frac{2}{9}$

Exercise:

Problem: $\frac{5}{12}\left(-\frac{8}{15}\right)$

Exercise:

Problem: $\left(-\frac{14}{15}\right)\left(\frac{9}{20}\right)$

Solution:

$-\frac{21}{50}$

Exercise:

Problem: $\left(-\frac{9}{10}\right)\left(\frac{25}{33}\right)$

Exercise:

Problem: $\left(-\frac{63}{84}\right)\left(-\frac{44}{90}\right)$

Solution:

$\frac{11}{30}$

Exercise:

Problem: $\left(-\frac{33}{60}\right)\left(-\frac{40}{88}\right)$

Exercise:

Problem: $4 \cdot \frac{5}{11}$

Solution:

$\frac{20}{11}$

Exercise:

Problem: $5 \cdot \frac{8}{3}$

Exercise:

Problem: $\frac{3}{7} \cdot 21n$

Solution:

$9n$

Exercise:

Problem: $\frac{5}{6} \cdot 30m$

Exercise:

Problem: $-28p\left(-\frac{1}{4}\right)$

Solution:

$7p$

Exercise:

Problem: $-51q\left(-\frac{1}{3}\right)$

Exercise:

Problem: $-8\left(\frac{17}{4}\right)$

Solution:

-34

Exercise:

Problem: $\frac{14}{5}(-15)$

Exercise:

Problem: $-1\left(-\frac{3}{8}\right)$

Solution:

$\frac{3}{8}$

Exercise:

Problem: $(-1)\left(-\frac{6}{7}\right)$

Exercise:

Problem: $\left(\frac{2}{3}\right)^3$

Solution:

$\frac{8}{27}$

Exercise:

Problem: $\left(\frac{4}{5}\right)^2$

Exercise:

Problem: $\left(\frac{6}{5}\right)^4$

Solution:

$\frac{1296}{625}$

Exercise:

Problem: $\left(\frac{4}{7}\right)^4$

Find Reciprocals

In the following exercises, find the reciprocal.

Exercise:

Problem: $\frac{3}{4}$

Solution:

$\frac{4}{3}$

Exercise:

Problem: $\frac{2}{3}$

Exercise:

Problem: $-\frac{5}{17}$

Solution:

$-\frac{17}{5}$

Exercise:

Problem: $-\frac{6}{19}$

Exercise:

Problem: $\frac{11}{8}$

Solution:

$\frac{8}{11}$

Exercise:

Problem: -13

Exercise:

Problem: -19

Solution:

$-\frac{1}{19}$

Exercise:

Problem: -1

Exercise:

Problem: 1

Solution:

1

Exercise:

Problem: Fill in the chart.

	Opposite	Absolute Value	Reciprocal
$-\frac{7}{11}$			
$\frac{4}{5}$			
$\frac{10}{7}$			
-8			

Exercise:

Problem: Fill in the chart.

	Opposite	Absolute Value	Reciprocal
$-\frac{3}{13}$			
$\frac{9}{14}$			
$\frac{15}{7}$			

	Opposite	Absolute Value	Reciprocal
-9			

Solution:

Number	Opposite	Absolute Value	Reciprocal
$-\frac{3}{13}$	$\frac{3}{13}$	$\frac{3}{13}$	$-\frac{13}{3}$
$\frac{9}{14}$	$-\frac{9}{14}$	$\frac{9}{14}$	$\frac{14}{9}$
$\frac{15}{7}$	$-\frac{15}{7}$	$\frac{15}{7}$	$\frac{7}{15}$
-9	9	9	$-\frac{1}{9}$

Divide Fractions

In the following exercises, model each fraction division.

Exercise:

Problem: $\frac{1}{2} \div \frac{1}{4}$

Exercise:

Problem: $\frac{1}{2} \div \frac{1}{8}$

Solution:

4

Exercise:

Problem: $2 \div \frac{1}{5}$

Exercise:

Problem: $3 \div \frac{1}{4}$

Solution:

12

In the following exercises, divide, and write the answer in simplified form.
Exercise:

Problem: $\frac{1}{2} \div \frac{1}{4}$

Exercise:

Problem: $\frac{1}{2} \div \frac{1}{8}$

Solution:

4

Exercise:

Problem: $\frac{3}{4} \div \frac{2}{3}$

Exercise:

Problem: $\frac{4}{5} \div \frac{3}{4}$

Solution:

$\frac{16}{15}$

Exercise:

Problem: $-\frac{4}{5} \div \frac{4}{7}$

Exercise:

Problem: $-\frac{3}{4} \div \frac{3}{5}$

Solution:

$-\frac{5}{4}$

Exercise:

Problem: $-\frac{7}{9} \div \left(-\frac{7}{9}\right)$

Exercise:

Problem: $-\frac{5}{6} \div \left(-\frac{5}{6}\right)$

Solution:

1

Exercise:

Problem: $\frac{3}{4} \div \frac{x}{11}$

Exercise:

Problem: $\frac{2}{5} \div \frac{y}{9}$

Solution:

$\frac{18}{5y}$

Exercise:

Problem: $\frac{5}{8} \div \frac{a}{10}$

Exercise:

Problem: $\frac{5}{6} \div \frac{c}{15}$

Solution:

$\frac{25}{2c}$

Exercise:

Problem: $\frac{5}{18} \div \left(-\frac{15}{24}\right)$

Exercise:

Problem: $\frac{7}{18} \div \left(-\frac{14}{27}\right)$

Solution:

$-\frac{3}{4}$

Exercise:

Problem: $\frac{7p}{12} \div \frac{21p}{8}$

Exercise:

Problem: $\frac{5q}{12} \div \frac{15q}{8}$

Solution:

$\frac{2}{9}$

Exercise:

Problem: $\frac{8u}{15} \div \frac{12v}{25}$

Exercise:

Problem: $\frac{12r}{25} \div \frac{18s}{35}$

Solution:

$\frac{14r}{15s}$

Exercise:

Problem: $-5 \div \frac{1}{2}$

Exercise:

Problem: $-3 \div \frac{1}{4}$

Solution:

-12

Exercise:

Problem: $\frac{3}{4} \div (-12)$

Exercise:

Problem: $\frac{2}{5} \div (-10)$

Solution:

$-\frac{1}{25}$

Exercise:

Problem: $-18 \div \left(-\frac{9}{2}\right)$

Exercise:

Problem: $-15 \div \left(-\frac{5}{3}\right)$

Solution:

9

Exercise:

Problem: $\frac{1}{2} \div \left(-\frac{3}{4}\right) \div \frac{7}{8}$

Exercise:

Problem: $\frac{11}{2} \div \frac{7}{8} \cdot \frac{2}{11}$

Solution:

$\frac{8}{7}$

Identify and Use Fraction Operations

In the following exercises, perform the indicated operations. Write your answers in simplified form.

Exercise:

Problem:

ⓐ $\frac{3}{4} + \frac{1}{6}$
ⓑ $\frac{3}{4} \div \frac{1}{6}$

Exercise:

Problem:

ⓐ $\frac{2}{3} + \frac{1}{6}$
ⓑ $\frac{2}{3} \div \frac{1}{6}$

Solution:

ⓐ $\frac{5}{6}$
ⓑ 4

Exercise:

Problem:

ⓐ $-\frac{2}{5}-\frac{1}{8}$
ⓑ $-\frac{2}{5}\cdot\frac{1}{8}$

Exercise:

Problem:

ⓐ $-\frac{4}{5}-\frac{1}{8}$
ⓑ $-\frac{4}{5}\cdot\frac{1}{8}$

Solution:

ⓐ $-\frac{37}{40}$
ⓑ $-\frac{1}{10}$

Exercise:

Problem:

ⓐ mfrac $\div\frac{8}{15}$
ⓑ mfrac $-\frac{8}{15}$

Exercise:

Problem:

ⓐmfrac $\div \frac{7}{12}$
ⓑmfrac $- \frac{7}{12}$

Solution:

ⓐ$\frac{9a}{14}$
ⓑ$\frac{9a-14}{24}$

Exercise:

Problem:

ⓐ$\frac{9}{10} \cdot ($−mfrac$)$
ⓑ$\frac{9}{10} + ($−mfrac$)$

Exercise:

Problem:

ⓐ$\frac{4}{15} \cdot ($−mfrac$)$
ⓑ$\frac{4}{15} + ($−mfrac$)$

Solution:

ⓐ$-\frac{4}{3q}$
ⓑ$\frac{12-25q}{45}$

Exercise:

Problem: $-\frac{3}{8} \div \left(-\frac{3}{10}\right)$

Exercise:

Problem: $-\frac{5}{12} \div \left(-\frac{5}{9}\right)$

Exercise:

Problem: $-\frac{3}{8} + \frac{5}{12}$

Exercise:

Problem: $-\frac{1}{8} + \frac{7}{12}$

Solution:

$\frac{11}{24}$

Exercise:

Problem: $\frac{5}{6} - \frac{1}{9}$

Exercise:

Problem: $\frac{5}{9} - \frac{1}{6}$

Solution:

$\frac{7}{18}$

Exercise:

Problem: $\frac{3}{8} \cdot \left(-\frac{10}{21}\right)$

Exercise:

Problem: $\frac{7}{12} \cdot \left(-\frac{8}{35}\right)$

Solution:

$-\frac{2}{15}$

Exercise:

Problem: $-\frac{7}{15}-\frac{y}{4}$

Exercise:

Problem: $-\frac{3}{8}-\frac{x}{11}$

Solution:

$\frac{-33-8x}{88}$

Exercise:

Problem: $\frac{11}{12a}\cdot\frac{9a}{16}$

Exercise:

Problem: $\frac{10y}{13}\cdot\frac{8}{15y}$

Solution:

$\frac{16}{39}$

Everyday Math

Exercise:

Problem:

Baking A recipe for chocolate chip cookies calls for $\frac{3}{4}$ cup brown sugar. Imelda wants to double the recipe.

ⓐ How much brown sugar will Imelda need? Show your calculation. Write your result as an improper fraction and as a mixed number.

ⓑ Measuring cups usually come in sets of $\frac{1}{8}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$, and 1 cup. Draw a diagram to show two different ways that Imelda could measure the brown sugar needed to double the recipe.

Exercise:

Problem:

Baking Nina is making 4 pans of fudge to serve after a music recital. For each pan, she needs $\frac{2}{3}$ cup of condensed milk.

ⓐ How much condensed milk will Nina need? Show your calculation. Write your result as an improper fraction and as a mixed number.
ⓑ Measuring cups usually come in sets of $\frac{1}{8}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$, and 1 cup. Draw a diagram to show two different ways that Nina could measure the condensed milk she needs.

Solution:

- ⓐ $4\frac{2}{3} = \frac{8}{3} = 2\frac{2}{3}$
- ⓑ Answers will vary.

Exercise:

Problem:

Portions Don purchased a bulk package of candy that weighs 5 pounds. He wants to sell the candy in little bags that hold $\frac{1}{4}$ pound. How many little bags of candy can he fill from the bulk package?

Exercise:

Problem:

Portions Kristen has $\frac{3}{4}$ yards of ribbon. She wants to cut it into equal parts to make hair ribbons for her daughter's 6 dolls. How long will each doll's hair ribbon be?

Solution:

$\frac{1}{8}\text{ yard}$

Writing Exercises

Exercise:

Problem: Explain how you find the reciprocal of a fraction.

Exercise:

Problem: Explain how you find the reciprocal of a negative fraction.

Solution:

Answers will vary.

Exercise:

Problem:

Give an example from everyday life that demonstrates how $\frac{1}{2}\cdot\frac{2}{3}\text{ is }\frac{1}{3}$.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
simplify fractions.			
multiply fractions.			
find reciprocals.			
divide fractions.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

reciprocal
: The reciprocal of the fraction $\frac{a}{b}$ is $\frac{b}{a}$ where $a \neq 0$ and $b \neq 0$.